

A Poisson Point Process Model for Coverage Analysis of Multi-hop Cooperative Networks

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Abstract—A spatial Poisson point process model for a multi-hop cooperative network with fixed hop boundaries and random number of the nodes in each hop is developed and analyzed. Transmissions over multi-hop network follow a decode-and-forward (DF) protocol that form Opportunistic Large Array (OLA). The probability density function (PDF) of the received power at a node is derived, when all the DF nodes in the previous level transmit with the same power in the presence of Rayleigh fading and path loss. The performance of the system is characterized in terms of one-hop success probability, which is calculated using the PDF of the received power at a node. The coverage range of the multi-hop network is analyzed for different set of network parameters.

Keywords—Strip-shaped networks, Opportunistic Large Array (OLA), Poisson point process, outage probability.

I. INTRODUCTION

Wireless networks are subject to multipath fading, which renders reliable communication difficult. Cooperation is a way to combat multipath fading and provides reliable communication [1]. Cooperation is achieved in such a way that the source has multiple independent fading paths to the destination owing to the broadcast nature of wireless channels, which results in multiple copies of the same signal to be received at the receiver forming a virtual multiple-input single-output (MISO). Spatial diversity can be achieved by applying combining techniques on these multiple copies of the same signal to achieve higher signal-to-noise ratio (SNR). It can also help in reaching distant destination without draining the entire power of the source, which will otherwise be drained quickly, thanks to diversity gain. In spatial diversity, an antenna array is to be established, whereas in cooperative diversity, a virtual array is established with the help of cooperating nodes.

Cooperative network involving multiple relays known as Opportunistic Large Array (OLA) [2], is a very promising technique for flooding or broadcasting the data effectively, as it prevents energy drainage of the source. A promising feature of this technique is that it does not require any addressing scheme or any prior location information of the nodes. While exploiting these features, OLA is a scalable technique. In simple OLA, every other node that receives source transmission, retransmits that signal to the other nodes in the vicinity, if

the received SNR is greater than the decoding threshold and node has not forwarded the same signal before. This process continues in a multi-hop fashion until the destination receives the data or the data is broadcasted to the entire network.

In this paper, we analyze the randomness of the nodes in a strip-shaped network and stochastically model the cooperative communication in between the random number of nodes, which takes place over multiple hops. The motivation behind this analysis is that if the nodes are randomly distributed between a source-destination pair, then how does cooperation works in between them in a multi-hop manner. Different OLA protocols like OLACRA [3], OLAROAD [4] and CBR [5] can be helpful in constructing such path. Combining multiple strip-shaped networks form a larger network, which can be used for different applications like performing structural health checks of tunnels and bridges [6] and setting up a communication system between vehicles on the highways [7].

The authors in [8] modeled a strip-shaped network with infinite node density and presented their results in terms of the successful signal reception at a distant receiver. Their results, however, cannot be applied to the network with lesser node density [9]. It is also shown in [10] that the infinite propagation is forbidden in the case of the finite node density. The authors in [11] and [12] showed that for the finite node density, infinite propagation is not possible, however data can be propagated to a destination node using the notion of success probability. OLA has become a very rich field and researchers have analyzed this field through various perspectives [13]–[17]. The authors in [18] modeled a linear network and then results were presented for strip-shaped network with fixed number of nodes in each level in [19].

The model presented in [19] successfully cater the problem of random locations of nodes in each level but their main restriction was the fixed number of nodes in each level. We, on the other hand, consider the random number of nodes along with random locations in each level of a strip-shaped network forming a spatial Poisson point process (PPP) and calculate the received power at a node in the next level when there are random number of decode-and-forward (DF) nodes in the previous level. Modeling random number of nodes is quite a difficult task because it does not form quasi-stationary markov chain as used in the case of the fixed number of nodes in [20]–[22]. A closed-form expression has been derived for the one-hop success probability and the outage probability of the node in the next level considering the underlying PPP model. Rayleigh fading and path loss are considered as channel impairments while deriving these closed-formed expressions.

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A useful coverage analysis is also done while utilizing these probabilities.

The rest of the paper is organized as follows. Section II describes the system model and mathematical preliminaries about the system. In Section III, a closed-form expression for the outage probability is derived followed by one-hop success probability. Section IV presents the results and validate the mathematical model followed by conclusion and future direction in Section V.

II. SYSTEM MODEL

This section presents the system model and assumptions for analyzing the random cooperative network. Consider a strip-shaped extended network with finite node density as shown in Fig. 1. The network is divided into hypothetical boundaries, which form multiple levels of square shape. Let ϕ denotes a stationary Poisson point process (PPP) with intensity (or density) $\tilde{\lambda}$ such that the number of nodes in each square $S \in \mathbb{R}^2$ is $\phi(S) = k$, where k is a Poisson random variable (RV) with mean $\gamma = \tilde{\lambda}|S|$ and its probability is given by

$$\mathbb{P}(\phi(S) = k) = \exp(-\tilde{\lambda}|S|) \frac{(\tilde{\lambda}|S|)^k}{k!}, \quad (1)$$

where $|S|$ is the area of square $L \times L$. Hence each square region contains a random number of nodes having random node locations. The nodes at level 1 receive the signal from some source node and retransmit the signal to level 2 at the next time slot if they successfully decode by using a DF mechanism. Each node in level 2 applies diversity combining on the multiple received copies of the same signal. If the received post-detection SNR at a node is greater than a predefined threshold, τ , then the node becomes a DF node. For retransmission, two assumptions are considered, i.e., the transmit power P_t of all the nodes is the same and secondly all the transmissions in one level are synchronized [23]. This is how cooperation is done for transmitting source node's signal towards the destination. Based on the threshold, τ , there are two types of nodes in each level, i.e., DF nodes whose SNR is greater than τ , represented as dark circles in Fig. 2 and nodes, which failed to decode the data represented as hollow circles. The density of the DF nodes is $\lambda = \tilde{\lambda}/P_s$ [24], where P_s is the probability that the received power is greater than the predefined threshold, τ . DF nodes follow the stationary PPP with reduced intensity λ . Transmission over one hop would be successful if there is at least one DF node in the next level. In this analysis, we consider two channel impairments namely the path loss and Rayleigh fading.

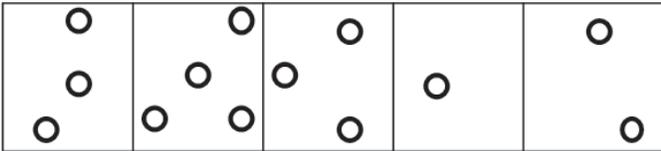


Fig. 1. A strip-shaped network with hypothetical boundaries and random node locations.

Let the number of DF nodes at a level n is given by $\phi(S_n)$, then the received power at a node j at level $n+1$ will be sum

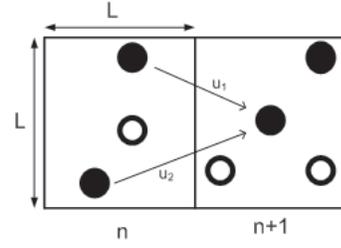


Fig. 2. A strip network with fixed boundaries and random number of nodes.

of powers received from all the DF nodes of previous level n given as

$$P_{r_j}(n+1) = \sum_{i \in \phi(S_n)} \frac{P_t \mu_{ij}}{d_{ij}^\alpha}. \quad (2)$$

In the above equation, μ_{ij} is the squared envelope of signal characterizing Rayleigh fading, d_{ij} is the Euclidean distance from node i to node j considering a 2D network, α characterizes path loss exponent and $\phi(S_n)$ denotes the number of nodes in the previous level. The squared envelope μ_{ij} is an exponential RV with unit mean whereas d_{ij}^α can be approximated as a Weibull RV with shape parameter $c = 3.1612/\alpha$ and scale parameter $\chi = (4L^2/3\Gamma(1.6327))^\alpha$ [25], where $\Gamma(\cdot)$ is an incomplete gamma function.

Analyzing this network requires the coverage probability, $P_s = \mathbb{P}(P_{r_j} > \tau)$, to be calculated. For coverage probability we need to find the PDF of the received power which in turn is a summation over a PPP of ratio of an exponential RV to a Weibull RV, which is derived in the next section.

III. OUTAGE PROBABILITY

It can be noticed that because of distance distribution, the outage probability of all receiving nodes is the same. Therefore, to find the PDF of the received power in (2), we need to determine the PDF of the ratio of exponential RV and Weibull RV, which is necessarily the distribution of received power from one transmitting node. However, since many nodes can transmit from the previous level, we need to self-convolve this distribution over the PPP. The authors in [19] have derived the distribution of the ratio of two variables, i.e. $R = \mu/d^\alpha$ (for simplicity, the subscripts are omitted that indicate node indices), which is given by

for $c \geq 1$ ($\alpha \leq 3.1612$)

$$f_R(r) = \chi \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma\left(\frac{1+c+n}{c}\right) (-r\chi)^n. \quad (3)$$

We now consider the following theorem that derives the outage probability of a node having received power as given in (2).

Theorem 1 (Outage Probability of a virtual MISO Network). *If nodes are placed according to a PPP with density $\tilde{\lambda}$ in adjoining square regions of $L \times L$, and the DF nodes with density λ at a previous level n transmit the signal to the nodes of the next level, then the outage probability, P_o , for an arbitrary node in the next level with $\alpha \leq 3.1612$ is given by*

$$P_o = \exp(-\lambda|S|) \left[\sum_{m=0}^{\infty} \frac{(\lambda|S|\chi)^m}{m!} \sum_{a_1=0}^{\infty} \sum_{a_2=0}^{\infty} \dots \sum_{a_m=0}^{\infty} \frac{(\tau/P_t)^{a_1+a_2+\dots+a_m+m}}{(a_1+a_2+\dots+a_m+m)!} \prod_{i=1}^m \Gamma\left(\frac{1+c+a_i}{c}\right) (-\chi)^{a_i} \right]. \quad (4)$$

$$f_{P_r}(p_r) = \exp(-\lambda|S|) \left[\sum_{m=0}^{\infty} \frac{(\lambda|S|\chi)^m}{m!} \sum_{a_1=0}^{\infty} \sum_{a_2=0}^{\infty} \dots \sum_{a_m=0}^{\infty} \frac{(p_r)^{a_1+a_2+\dots+a_m+m-1}}{(a_1+a_2+\dots+a_m+m-1)!} \prod_{i=1}^m \Gamma\left(\frac{1+c+a_i}{c}\right) (-\chi)^{a_i} \right]. \quad (11)$$

Proof: Assuming independent fading on each link, the ratio RVs are independent and identically distributed (I.I.D), hence the self-convolution of the PDFs of these RVs is equal to the product of their Laplace transforms, given as

$$\mathcal{L} [*_z^k f_{R_z}(r_z)] = \prod_{z=1}^k F_z(s) = (F(s))^k, \quad (5)$$

where $*$ represents the convolution operator, \mathcal{L} denotes the Laplace operator and $F(s)$ is given as

$$F(s) = \chi \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma\left(\frac{1+c+n}{c}\right) (-\chi)^n \frac{1}{s^{n+1}}. \quad (6)$$

In (5), k is the number of the transmitters (DF nodes) in the previous level. Since k is a Poisson RV, hence we have to find the expectation of $(F(s))^k$ with respect to k with mean $\gamma = \lambda|S|$, which is given as

$$G(s) = \mathbb{E} [(F(s))^k] = \sum_{k=0}^{\infty} \frac{(\lambda|S|)^k \exp(-\lambda|S|)}{k!} (F(s))^k. \quad (7)$$

After some mathematical manipulation, we obtain

$$G(s) = \exp\left(\lambda|S|\chi \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma\left(\frac{1+c+n}{c}\right) (-\chi)^n \frac{1}{s^{n+1}}\right) \times \exp(-\lambda|S|). \quad (8)$$

A direct inverse Laplace transform of the above equation is prohibited in closed-form, hence expanding the first exponential function in (8) with Taylor series, we get

$$G(s) = \sum_{m=0}^{\infty} \left(\lambda|S|\chi \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma\left(\frac{1+c+n}{c}\right) (-\chi)^n \frac{1}{s^{n+1}} \right)^m \times \exp(-\lambda|S|). \quad (9)$$

Solving the above equation for m , we get

$$G(s) = \left[\sum_{m=0}^{\infty} \frac{(\lambda S \chi)^m}{m!} \sum_{a_1=0}^{\infty} \dots \sum_{a_m=0}^{\infty} \frac{1}{s^{a_1+a_2+\dots+a_m+m}} \prod_{i=1}^m \Gamma\left(\frac{1+c+a_i}{c}\right) (-\chi)^{a_i} \right] \exp(-\lambda|S|). \quad (10)$$

The inverse Laplace transform of the above equation provides the PDF of the received power, P_r , which is given as

To obtain the CDF, $F_{P_r}(p_r)$, of the received power we integrate the above equation and evaluate it at $F_{P_r}(\tau/P_t)$ to get the outage probability P_o as given in (4). ■

Theorem 1 provides the outage probability of a node in the next level when the received power is a random sum of random variables. Outage probability is inversely proportional to density of the DF nodes and directly proportional to τ/P_t . Outage probability is also directly proportional to χ and χ is directly proportional to L , so outage probability is directly proportional to L . This outage probability is the same for each node of the next level. Therefore, to find the one-hop success, there must be at least one DF node in the next level. Using the outage probability, P_o , from (4) we can calculate the one hop success probability, ρ , which is given as

$$\begin{aligned} \rho &= \mathbb{P}(k \geq 1) \\ &= 1 - \mathbb{P}(k = 0) \\ &= 1 - \exp(\lambda|S|(1 - P_o)). \end{aligned} \quad (12)$$

IV. RESULTS AND DISCUSSION

In this section, we validate our analytical model by comparing it with numerical simulations and characterize the system performance in terms of the coverage range and examine the effects of node density and hop length on the coverage. For the sake of simulation results, we take two contiguous squares of length L and distribute nodes using a Poisson RV and calculate the received power at a node in the square using (2). We repeat this process for over 100,000 iterations and calculate the outage probability by comparing the received power with a certain threshold, τ , when the transmit power of all the nodes is same $P_t = 1$.

Fig. 3 validates the outage probability expression in (4) to that of simulation. To calculate outage probability, received power is compared with a threshold, τ , for 100,000 iterations. Three different nodes intensities 0.75, 1 and 1.25 are considered for validation of outage probability expression derived in Theorem 1. All three results match closely with the simulation results. The outage probability increases as τ increases but the increase in outage probability is significant as the node density is decreased from 1 to 0.75. Analytical expression involves infinite summation but accuracy up-to the two decimal points can be achieved from first six terms of the summation.

Fig. 4 shows the one-hop success probability, ρ , of the random network with different finite node densities as a function of the SNR margin, $\psi = P_t/\tau$. The length of each square region is $L = 5$ and path loss exponent is $\alpha = 2$. From node density, λ , and area of the square region, $|S|$, average number of the nodes in each case is calculated. The average number of the nodes are 2, 3 and 4 for node densities 0.08, 0.12

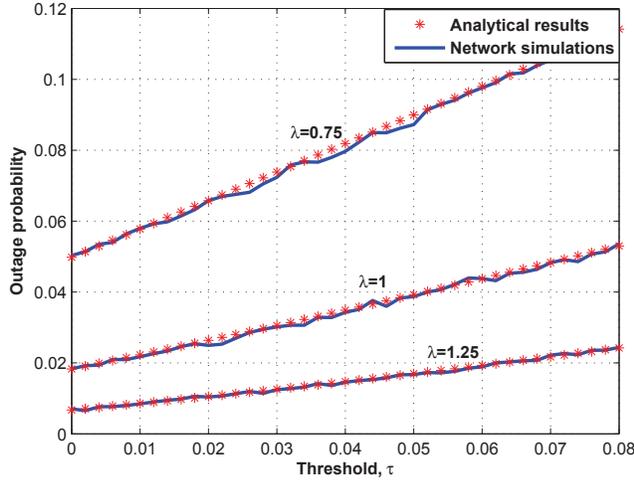


Fig. 3. Outage probability with $P_t = 1$, $L = 2$ and $\alpha = 2$.

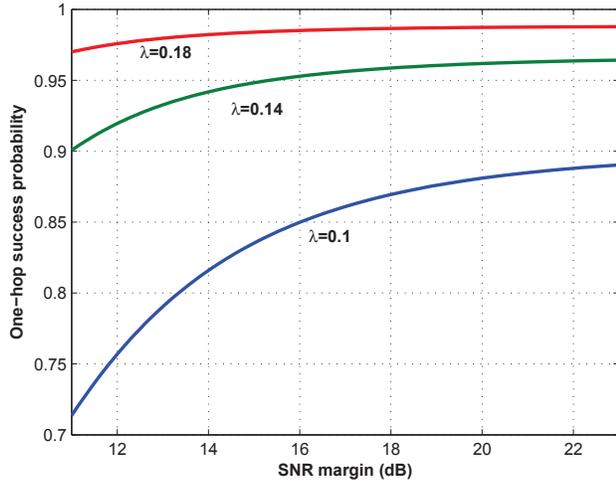


Fig. 4. One-hop success probability with $L = 5$ and $\alpha = 2$.

and 0.16, respectively. One-hop success probability increases with the increasing SNR margin for a fixed node density and it also increases as node density increases for a fixed SNR margin. From the results, it is quite evident that the one-hop success probability, ρ , increases significantly as the average number of nodes increases from 2 to 3. When the average number of nodes is increased to 4, one-hop success probability remain above 90% even for a small SNR margin of 10dB.

In case of the practical deployment of the network, the performance of the system can be characterized by finding the m -hop success probability or by finding the coverage range (CR) through which the signal propagates observing a certain quality of service (QoS), η . QoS defines the required end-to-end probability of success. If the signal traverses m hops, then the m -hop success probability is given by

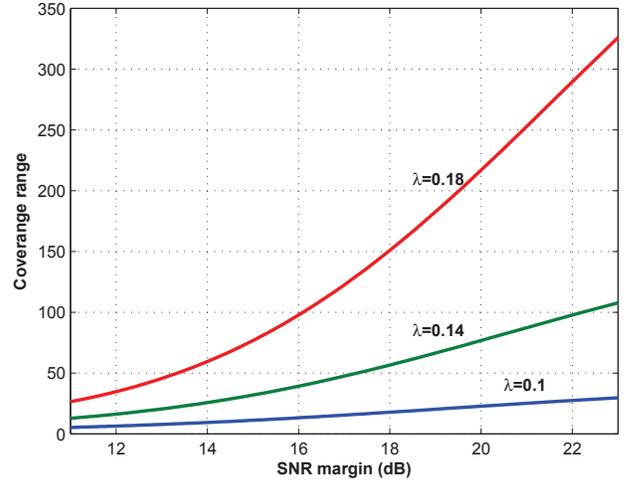


Fig. 5. Coverage range with $\eta = 0.7$, $L = 5$ and $\alpha = 2$.

$$\mathbb{P}(k \geq 1) = 1 - \exp(-\lambda |S| P_s^m). \quad (13)$$

For finding the number of hops which the signal traverses, η acts as an upper bound on m -hop success probability, i.e., $\mathbb{P}(k \geq 1) \geq \eta$. The network and transmission conditions (α , L , P_t , etc.) define the value of P_s and comparing that value with η , we calculate the number of hops, m , given as

$$m \leq \frac{\ln[-\ln(1-\eta)/(\lambda|S|)]}{\ln P_s}. \quad (14)$$

The average value of CR is calculated as

$$CR = mL. \quad (15)$$

Fig. 5 shows the CR for different node densities as a function of SNR margin, ψ , while $L = 5$, $\alpha = 2$ and $\eta = 0.7$. Increasing the node density increases the CR but this increase is significant when λ is increased from 0.12 to 0.16 as CR is increased from 60 to 190 for 23dB SNR margin. Increase in the SNR margin has different impact on the CR for different node densities. For $\lambda = 0.08$, CR is improved by 75% as SNR margin is increased from 11dB to 23dB, whereas for $\lambda = 0.16$, the CR is improved by 965% for same change in the SNR margin.

The contours of CR are plotted against the region length L and node density λ , shown in Fig. 6. The contour plot determines that the same CR can be achieved for different set of intensity λ and region length L . This enables network designer to achieve required CR while choosing the optimal λ and L according to the network requirements. These results are calculated for $\eta = 0.7$ and $\psi = 14dB$. The CR can be increased by keeping the region length small and increasing the node density. The increase in the region length, L , increases the path loss due to which the CR decreases, while keeping the λ and ψ constant.

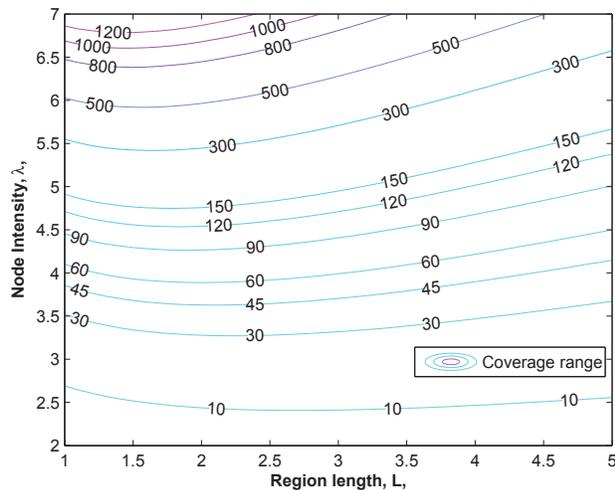


Fig. 6. Coverage range for different sets of region length L and intensity λ , with $\eta = 0.7$ and $\psi = 14\text{dB}$.

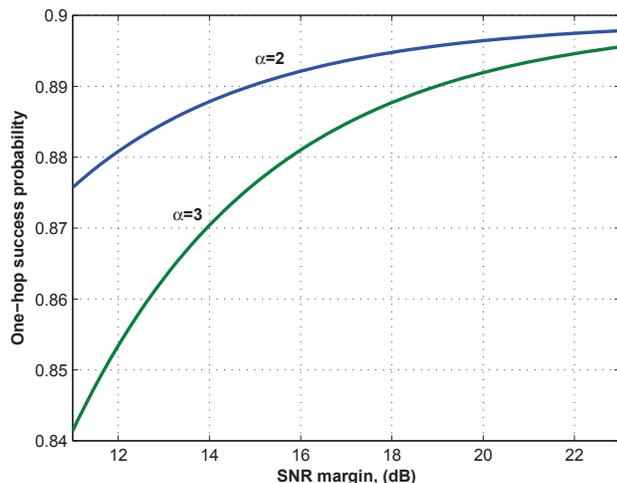


Fig. 7. Effect of path loss coefficient, α , on one-hop success probability with $\lambda = 0.62$ and $L = 2$.

Fig. 7 shows the effect of the path loss coefficient, α , on the one-hop success probability for $\lambda = 0.62$ and $L = 2$. The decrease in the one-hop success probability by increasing α indicates that the same one-hop success probability can be achieved for higher values of α at higher SNR margin, ψ . The two curves of the one-hop success probability are very close to each other at higher SNR margin as compared to lower values.

V. CONCLUSION

In this paper, the transmissions over hypothetical boundaries in a strip-shaped network of finite node density that form OLA is modeled with a spatial Poisson point process. The PDF of the received power at a node is derived by determining the self convolution of the ratio distribution of exponential RV to Weibull RV over the PPP. Using the PDF of the received

power, the outage probabilities of the nodes in each level and one-hop success probability of the network are calculated. A future direction of this work would be to study such random network with irregular hop boundaries instead of fixed ones.

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