On the Ratio of Exponential and Generalized Gamma Random Variables with Applications to Ad Hoc SISO Networks

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***Abstract*—A Poisson point process (PPP)-based model for a single-input single-output (SISO) transmission between two randomly located nodes is developed and analyzed. The power received at a node, when a randomly deployed transmitter transmits the message signal in the presence of Rayleigh fading and path loss, is shown to be the ratio of an exponential random variable (RV) and a generalized gamma (GG) RV. The cumulative distribution function (CDF) of the received power is derived, which is used to find the outage probability at the receiver. The study is then further extended to SISO multi-hop links where the coverage probability of the network is calculated. Finally, a transmission model is proposed that saves a significant amount of energy by carefully selecting an intermediate node in a multi-hop network such that the lifetime of the network can be increased. Numerical simulations are presented to validate the theoretical models.**

Keywords*—Strip-shaped networks, single-input single-output (SISO), Poisson point process, outage probability, ratio of exponential and generalized gamma.

I. INTRODUCTION

In large-scale wireless ad hoc and sensor networks, a message signal traverses multiple hops to reach a far off destination. The signal path is generally composed of various point-to-point (P2P) links where each link experiences independent power loss and is established opportunistically. This provides a flexible communication regime where the nodes can join the network freely and can switch to any other network without any hindrance [1].

Although P2P links are easy to form, each link should be reliable enough to propagate the message further because the failure at one link essentially fails the entire system. The end-to-end success probability is the product of individual success probabilities of each link, which becomes sufficiently small as the signal propagates the network. Cooperative transmission (CT) becomes an efficient choice employed to increase the reliability of such links where multiple copies of the same signal are transmitted to provide diversity in the network [2]. CT is a special case of virtual multiple-input single-output (MISO) systems where distributed nodes provide spatial diversity to the receiver and each node forms an independent single-input single-output (SISO) link with the receiver. The CT in strip-shaped network is analyzed through various perspectives but with the assumption of the fixed hop boundary [3]–[5]. To remove this constraint, the study of SISO links with no fixed hop boundary is required. However, to analytically model a SISO link with no fixed boundary, the distribution of the received signal-to-noise ratio (SNR) is required, which takes the form of a ratio of two random variables (RVs) when random path loss and channel fading are under consideration. Hence, the motivation behind this study is to derive a closed-form expression of the outage probability for a SISO link with above channel impairments, which eventually can be used to model more general cooperative networks.

To study the characteristics of transmissions over an ad-hoc network, where the nodes are distributed uniformly in a region, the so-called Poisson point process (PPP) model is generally a good choice [6], [7]. In a Poisson point field, the random distance between a node and its nth neighbor is given by the generalized gamma (GG) distribution [8]. This, in turns, makes the notion of path loss to be modeled by a GG RV. In the presence of Rayleigh fading, the received power for a SISO link is the ratio of exponential and GG RVs.

The authors in [9] introduced the concept of GG distribution and it got immense importance because of its use in multiple fields [10]–[13]. Several well-known distributions can be derived from GG distribution as special and limiting cases such as Rayleigh, lognormal, gamma, Weibull and Nakagami. A number of product and ratio distributions involving GG distribution have been derived in technical literature [14]–[16], but they assumed the same value of the shape parameter β while evaluating the product and ratio distributions, which limits their usefulness. Author in [17] have derived the product distribution of GG RVs with different β. However, to the best of author's knowledge, no literature has been found in which the problem of the ratio of an exponential and a GG RV has been addressed or the ratio distribution of the GG RVs with different shape parameters has been derived from which the ratio of an exponential and a GG RV could be deduced.

In this paper, the ratio distribution between the exponential and GG RVs is derived, which provides the expression for calculating the cumulative distribution function (CDF) of the received power and the outage probability for an ad hoc SISO link. This ratio distribution is then used in analyzing the wireless networks for the above-mentioned practical scenarios. Using the derived results, the coverage probability of a SISO multi-hop network can be obtained for a given quality-of-service (QoS). We also provide a way of node participation in the network to increase the network lifetime and to conserve the overall energy of the system.

The paper is oraganized as follows. In Section II, system model is presented along with preliminary distributions. In Section III, the distribution of the ratio is derived and the analytical expression for the coverage probability of the SISO network is also presented followed by the results and discussion in Section IV. The paper is concluded in Section V along with the future direction.

II. System Model

Consider a 2-dimensional (2D) strip-shaped network with randomly distributed finite number of nodes as shown in Fig. 1. Let Φ denotes a stationary Poisson point process (PPP) with

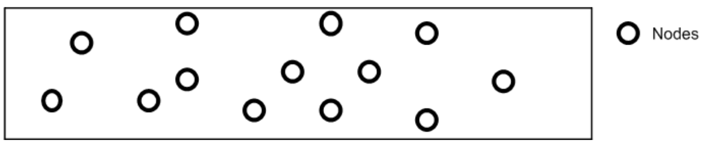

Fig. 1. A strip-shaped network with finite node density and random node locations.

density λ on a strip-shaped network of area $|A| \in \mathbb{R}^2$ such that the average number of nodes in the network is $\gamma = \lambda|A|$. The number of nodes in the network is given by $\Phi(A) = n$ with probability

$$\mathbb{P}(\Phi(A) = n) = \exp\left(-\lambda|A|\right)\frac{(\lambda|A|)^n}{n!}. \tag{1}$$

The message propagates to the destination in a single-input single-output (SISO) fashion over multiple hops. The communication between two random nodes is shown in Fig. 2. The received signal will be forwarded to the next node if the received signal power at the current node is greater than a predefined decoding threshold, τ. The received power while

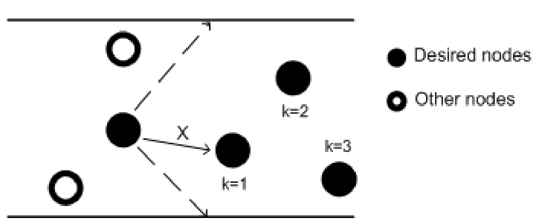

Fig. 2. Transmission of signal from one node to another without fixed boundary

considering only two channel distortions, fading and path loss, is given by

$$P_r = \frac{P_t X}{d^\alpha}, \tag{2}$$

where P_t is the transmit power, X characterizes the phenomenon of fading, d represents the Euclidean distance between the transmitter and the receiver and α is the path loss exponent. The RV X is drawn from an exponential distribution, which models the squared envelope of the signal experiencing Rayleigh fading. The probability density functions (PDF) of X is given by

$$f_X(x) = \mu \exp\left(-\mu x\right), \tag{3}$$

where μ is the mean of the exponential distribution. The distance, d, to the kth nearest neighbor is modeled by a GG RV [18], W, and its PDF is given by

$$f_W(w) = \frac{\chi}{\Gamma(k)\,\delta^{k\chi}} w^{k\chi-1} \exp\left\{-\left(\frac{w}{\delta}\right)^{\chi}\right\}, \tag{4}$$

where $\chi = 2$ characterizes a 2D network and $\Gamma(.)$ is the gamma function. In the above equation, $\delta = \left(\lambda c_{\phi,\chi}\right)^{-1/\chi}$ and $c_{\phi,\chi}$ is the parameter that determines the angular range where the next neighboring node lies. For efficient transmission, the next hop node should be nearer to the destination, i.e., the next hop node should lie within the specified angle $0 < \phi < \frac{\pi}{2}$ as reference to the source-destination vector. Usually $c_{\phi,\chi} = \pi/4$ for neighboring node to exist within $90°$ sector as shown in Fig. 2 with dotted lines. There may be many neighboring nodes to be part of next hop and the value of k determines that specific node to which the SISO link should be established. If $k = 1$, then node communicates with the first nearest neighbor. Whereas, when $k = 2$, the transmitting node communicates with the second nearest neighbor as shown in Fig. 2. The distance between the nodes is distributed according to (4), however, the distribution of the distance raised to power, α, where $d^\alpha \in Y$ and $Y = W^\alpha$ is given as

$$f_Y(y) = \frac{1}{\alpha} y^{\frac{1}{\alpha}-1} f_W(y^{\frac{1}{\alpha}}). \tag{5}$$

Using (4) and (5), the distribution of the distance raised to power α is a GG distribution with shape parameters k and $\beta = \chi/\alpha$, and scale parameter, $\theta = \delta^\alpha$, given by

$$f_Y(y) = \frac{\beta}{\Gamma(k)\,\theta^{k\beta}} y^{k\beta-1} \exp\left\{-\left(\frac{y}{\theta}\right)^{\beta}\right\}. \tag{6}$$

It can be observed from (2) that the coverage probability, P_s, of the receiving node is a doubly stochastic process that depends upon two RVs. For coverage probability, the CDF of the received power is required where the received power in (2) is the ratio of two RVs, i.e., exponential RV, X, and GG RV, Y. The ratio distribution of an exponential RV and GG RV is derived in the next section.

III. Ratio Distribution And Coverage Probability

The closed-form expression of the CDF of the ratio in (2) is expressed in the following theorem.

Theorem 1. *The CDF of $Z = X/Y$, where X and Y are distributed according to (3) and (6) respectively, is given by*

$$F_Z(z) = \frac{\beta}{\Gamma(k)\,\theta^{k\beta}} \left\{ \frac{\Gamma(k)}{\beta\theta^{-\beta k}} - (\mu z)^{-k\beta} \times H_{1,1}^{1,1}\left[(\theta\mu z)^{-\beta} \middle| \begin{matrix} (1-k\beta,\beta) \\ (0,1) \end{matrix} \right] \right\}, \tag{7}$$

where $H_{:,:}^{:,:}[.]$ represents the Fox H-function [19].

Proof: The CDF of Z involves the CDF of X given by

$$F_X(x) = 1 - \exp\left(-\mu x\right). \tag{8}$$

Using (6) and (8), the CDF of Z can be calculated as

$$\mathbb{P}(Z \leq z) = \int_0^\infty F_X(zy) f_Y(y)\, dy \tag{9}$$

$$= \frac{\beta}{\Gamma(k)\,\theta^{k\beta}} \int_0^\infty \left(1 - e^{-\mu z y}\right) y^{k\beta-1} e^{-\left(\frac{y}{\theta}\right)^\beta} dy \tag{10}$$

$$= \frac{\beta}{\Gamma(k)\,\theta^{k\beta}} \left[\int_0^\infty y^{k\beta-1} e^{-\left(\frac{y}{\theta}\right)^\beta} dy - \right. \tag{11}$$

$$\left. \int_0^\infty e^{-\mu z y} y^{k\beta-1} e^{-\left(\frac{y}{\theta}\right)^\beta} dy \right]$$

$$= \frac{\beta}{\Gamma(k)\,\theta^{k\beta}} [A - B], \tag{12}$$

where $A = \int_0^\infty y^{k\beta-1} e^{-\left(\frac{y}{\theta}\right)^\beta} dy$ and $B = \int_0^\infty e^{-\mu z y} y^{k\beta-1} e^{-\left(\frac{y}{\theta}\right)^\beta} dy$. A can be solved analytically after some mathematical manipulations as

$$A = \frac{\Gamma(k)}{\beta\theta^{-k\beta}}, \tag{13}$$

The two exponential functions in integral B can be represented in terms of the Fox H-functions using (2.9.4) of [19], given as

$$\exp(-\mu z y) = H_{0,1}^{1,0}\left[\mu z y \,\middle|\, \begin{matrix} - \\ (0,1) \end{matrix}\right], \tag{14}$$

$$\exp\left(-(y/\theta)^\beta\right) = H_{0,1}^{1,0}\left[(y/\theta)^\beta \,\middle|\, \begin{matrix} - \\ (0,1) \end{matrix}\right]. \tag{15}$$

Using the above relations, the integral B becomes the product of the power function and the two Fox H-functions, which can be solved using (2.8.4) of [19], given as

$$B = \int_0^\infty y^{k\beta-1} H_{0,1}^{1,0}\left[(y/\theta)^\beta \,\middle|\, \begin{matrix} - \\ (0,1) \end{matrix}\right] \times$$

$$H_{0,1}^{1,0}\left[\mu z y \,\middle|\, \begin{matrix} - \\ (0,1) \end{matrix}\right] dy \tag{16}$$

$$B = (\mu z)^{-k\beta} H_{1,1}^{1,1}\left[(\theta\mu z)^{-\beta} \,\middle|\, \begin{matrix} (1-k\beta, \beta) \\ (0,1) \end{matrix}\right].$$

Using (13) and (16) in (12), the CDF of Z becomes

$$\mathbb{P}(Z \leq z) = \frac{\beta}{\Gamma(k)\,\theta^{k\beta}} \left\{ \frac{\Gamma(k)}{\beta\theta^{-\beta k}} - (\mu z)^{-k\beta} \times \right.$$
$$\left. H_{1,1}^{1,1}\left[(\theta\mu z)^{-\beta} \,\middle|\, \begin{matrix} (1-k\beta, \beta) \\ (0,1) \end{matrix}\right] \right\}. \tag{17}$$

■

Corollary 1. *For $\beta = 1$ ($\alpha = 2$), the CDF of Z is given as*

$$\mathbb{P}(Z \leq z) = \frac{1}{\Gamma(k)\,\theta^{k}} \left\{ \frac{\Gamma(k)}{\theta^{-k}} - (\mu z)^{-k} \times \right.$$
$$\left. \frac{\Gamma(k)}{(1 + 1/(\theta\mu z))^k} \right\}. \tag{18}$$

The ratio distribution derived in Theorem 1 is the distribution of the received power, which can be further used to derive the coverage probability at a particular node. The coverage probability is given by

$$P_s = \mathbb{P}(P_r \geq \tau) = 1 - \mathbb{P}(P_r \leq \tau) = 1 - P_o, \tag{19}$$

where P_o is the outage probability. The expression for the outage probability at a node is derived by evaluating the CDF of the ratio, $F_Z(\tau/P_t)$, and is given by

$$P_o = \frac{\beta}{\Gamma(k)\,\theta^{k\beta}} \left\{ \frac{\Gamma(k)}{\beta\theta^{-\beta k}} - (\mu\tau/P_t)^{-k\beta} \times \right.$$
$$\left. H_{1,1}^{1,1}\left[(\theta\mu\tau/P_t)^{-\beta} \,\middle|\, \begin{matrix} (1-k\beta, \beta) \\ (0,1) \end{matrix}\right] \right\}. \tag{20}$$

Similarly for $\alpha = 2$, the simplified expression of the outage probability, P_o, is given by

$$P_o = \frac{1}{\Gamma(k)\,\theta^{k}} \left\{ \frac{\Gamma(k)}{\theta^{-k}} - (\mu\tau/P_t)^{-k} \times \right.$$
$$\left. \frac{\Gamma(k)}{(1 + 1/(\theta\mu\tau/P_t))^k} \right\}. \tag{21}$$

IV. Results and Discussion

In this section, we verify the results of Theorem 1 along with the performance characterization of the SISO network in terms of coverage probability, hop count and energy conservation. The analytical expression derived in the theorem is verified by comparing it with Monte-Carlo simulations. For simulation purpose, exponential and generalized gamma RVs are generated and their ratio is calculated using (2). This process is repeated over 100,000 iterations and the CDF is evaluated. The value of path loss exponent, α, controls the value of β as shown in (6). The three values of α are used, which subsequently provide three different values for β, i.e., $\beta = 0.8, \beta = 1$ and $\beta = 2$, respectively.

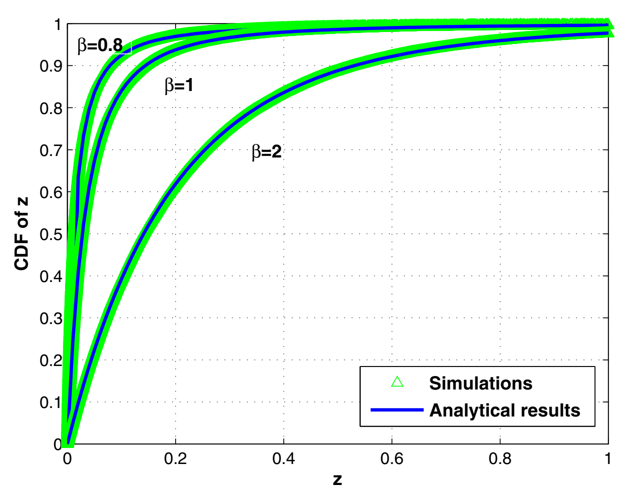

Fig. 3. CDF of the ratio for $k = 2$, $\theta = 15^\beta$ and $\mu = 1$.

Figs. 3 and 4 verify the ratio distribution obtained in Theorem 1. It can be seen that the analytical results closely match with the simulation results calculated for two different sets of parameters. For each set, three results are calculated and compared for three different conditions of β. Studying this graph help us understand the effects of parameters on the distribution. Fig. 3 shows that the change in the value of β, changes the shape parameter of the distribution. In Fig. 4, where k is set to unity, the CDF saturates early with the corresponding increase in the value of β, whereas in Fig. 3, where $k = 2$, the CDF saturates early with the corresponding decrease in the value of β. This behavior is due to change in the value of k, which is also a shape parameter of the GG distribution. The parameters k and β changes the shape of the ratio distribution. The effect of the scale parameter, θ, on the ratio distribution is somewhat opposite in nature, as it is evident from the figures that the CDF of the ratio scales back when the value of θ is increased from 1 to 15.

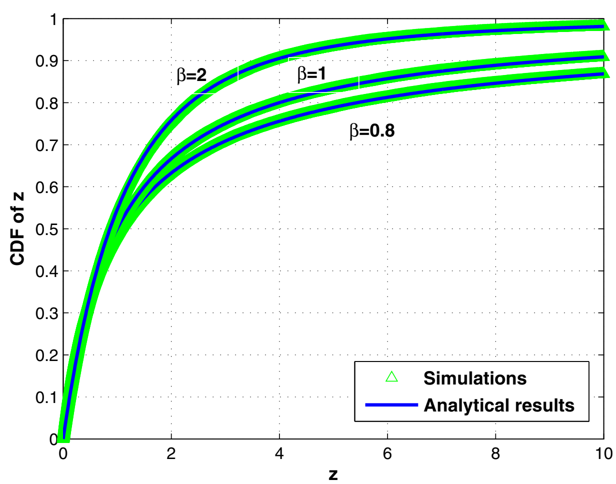

Fig. 4. CDF of the ratio for $k = 1$, $\theta = 1$ and $\mu = 1$.

Fig. 5 shows the trends of the coverage probability of a SISO link versus the SNR margin for various node densities. SNR margin is the normalized threshold defined as $\psi = P_t/\tau$. The effect of the SNR margin and node density on the coverage probability is quite evident. The coverage probability is directly proportional to the SNR margin as well as the node density. Keeping one variable fixed and increasing the other, increases the coverage probability. At low SNR margin, the density of the network plays an important role. For instance, it can be seen that at $5dB$, 16.7% increase in coverage probability is obtained by doubling the density of the network. Whereas, when λ is increased from 2 to 3, the corresponding increase in coverage probability is only 5.9%. Hence a trend of diminishing returns can be observed.

For the analysis of the multi-hop SISO network, m-hop success probability is an important parameter, defined as ${P_s}^m$, where P_s is the one-hop success probability. The m-hop success probability defines the end-to-end success probability of delivering the message to the mth hop. Generally it is desired that a quality of service (QoS), η, should be maintained in the network where η can be defined as the end-to-end success probability for m hops. Hence, ${P_s}^m \geq \eta$, to guarantee a successful transmission. An upper bound on the number of

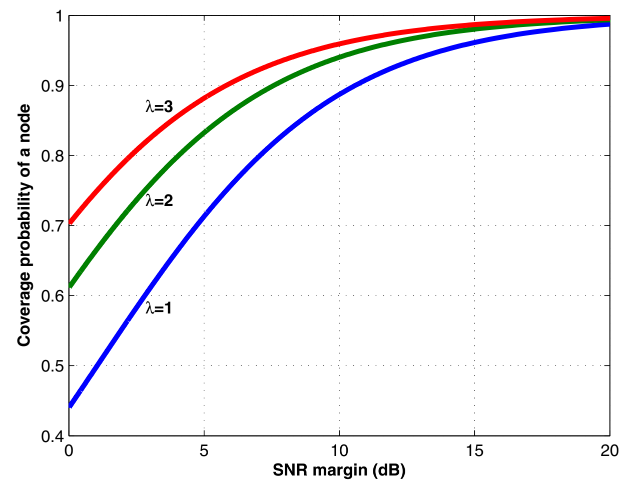

Fig. 5. Coverage probability of a SISO link for $k = 1$, $\mu = 1$, $\alpha = 2$ and $\beta = 1$.

hops can be evaluated given as

$$m \leq \frac{\ln \eta}{\ln P_s}. \tag{22}$$

Fig. 6 shows the maximum hop count of a SISO multi-hop network for a specific set of parameters plotted against the SNR margin. The maximum hop count increases significantly as SNR margin is increased above $25dB$. The maximum hop count also increases with the increasing node density. However, there is a trade off between these two parameters for achieving a specific maximum hop count. For instance, to achieve a hop count of 100, a network with node density of $\lambda = 2$ requires 13.9% less transmit power as compared to $\lambda = 1$ for fixed threshold level. Whereas, the transmit power requirement reduces by 21.8% for a network with a node intensity of $\lambda = 4$ and 17.5% for a node density of $\lambda = 3$, respectively, when compared with a density of $\lambda = 1$. Hence a suitable combination of these parameters can be used for the optimal performance of the multi-hop network.

Fig. 7 shows the coverage probability of the SISO network with limited node participation. As discussed previously, if $k = 1$ is used in (4), every other node communicates with its first neighboring node. Similarly, for $k = 2$, the node communicates with the second neighboring node as shown in Fig. 2 and the energy of the first neighboring node can be conserved. This model reduces the energy consumption by a factor of two as compared to $k = 1$, provided the coverage probability is above a defined QoS for a given SNR margin. If node communicates to the third neighboring node, i.e., $k = 3$, then, at maximum, $2/3$ of the energy of the system will be conserved as long as the coverage probability is above the limit for a given SNR margin. From Fig. 7, at an SNR margin of $11dB$, the coverage probability for both $k = 1$ and $k = 2$ are well above 80%, which is our desired QoS, so alternative nodes can be used for communication and almost 50% of the energy can be conserved. Similarly at an SNR margin of $23dB$, coverage probability for all three cases are almost 100%, so the system will be more energy efficient if $k = 3$ is used as

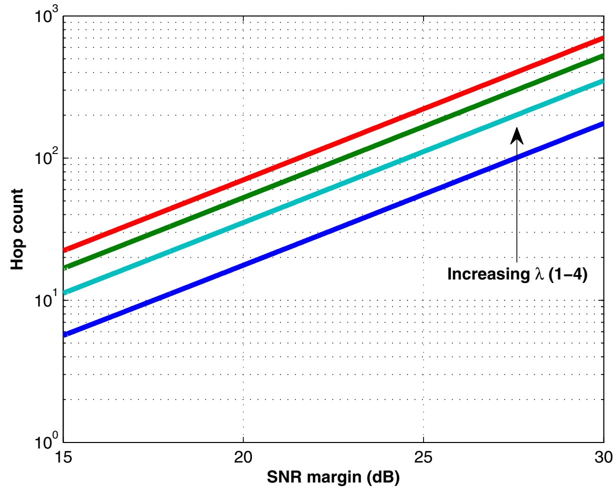

Fig. 6. Hop count for $k = 1$, $\mu = 1$, $\alpha = 2$, $\beta = 1$ and $\eta = 0.8$.

compared to other two values.

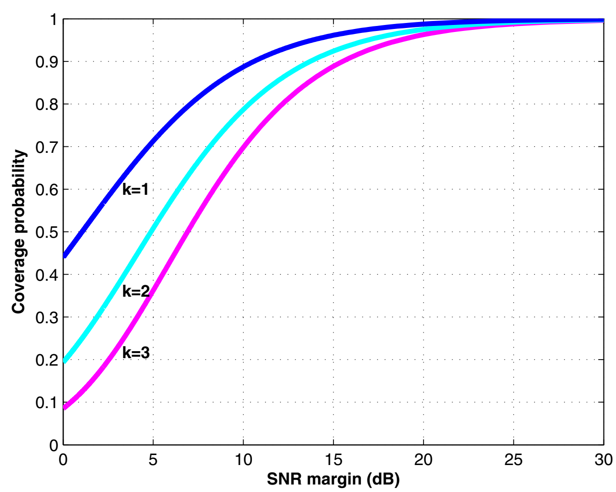

Fig. 7. Effect of nearest communicating node on the coverage probability for $\lambda = 1$, $\mu = 1$, $\alpha = 2$ and $\beta = 1$.

V. Conclusion

A stochastic model for performance evaluation of the SISO links experiencing Rayleigh fading in ad-hoc network is presented. The randomness of the nodes is modeled with PPP and GG distribution is used to model the distance. The ratio distribution of the exponential RV and GG RV has been derived. This distribution is used to determine the coverage probability of a node in SISO network and the reliability of the system. The coverage probability is used to find the hop count, which the signal traverses in a multi-hop environment. Energy efficiency of the model is studied when node participation is limited. A significant future direction of this work would be the use of this ratio distribution in finding the received power distribution of a multiple-input single-output (MISO) link.

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