A Stochastic Geometry Approach for Outage Analysis of Ad Hoc SISO Networks in Rayleigh Fading

Asma Afzal and Syed Ali Hassan
School of Electrical Engineering and Computer Science
National University of Sciences and Technology, Pakistan
Email: {asma.afzal, ali.hassan}@seecs.edu.pk

***Abstract*—In this paper, a closed-form expression for the outage probability of an ad hoc single-input single-output (SISO) network is derived when the network is subject to Rayleigh fading and arbitrary path loss model. The underlying stochastic geometry of the network assumes a simple point process in which a single node is placed randomly in a square region. A closed-form expression of the distribution of the distance between a pair of nodes is derived. This expression is then generalized for positive powers to incorporate the path loss exponent. Moment matching is used to make the model analytically tractable. It is shown that the distribution of the distance is well approximated by the Weibull distribution with the scale parameter depending on the size of the network and a constant shape parameter. The coverage and outage behaviors of the network for various network sizes and path loss exponents are quantified. Numerical simulations are performed to validate the theoretical models.**

***Keywords*—SISO networks, Weibull distribution, outage probability.**

I. Introduction

In wireless ad-hoc networks, message delivery usually takes place in a conventional multi-hop fashion between far off nodes. A point-to-point link is established between the nodes and the message traverses through multiple such links. However, because of the varying nature of wireless channel, if a single wireless link fails during transmission, the probability of successful delivery of the message at the destination approaches zero. Cooperative transmission (CT) was introduced to exploit the inherent features of a wireless channel by employing spatial diversity. In a CT network, multiple nodes transmit the same message, thereby providing transmit diversity. A node receiving signals arriving from multiple uncorrelated channels is more likely to decode the message than the node receiving signals from a single channel. Hence, CT increases the received signal-to-noise ratio (SNR) and consequently provides range extension. Cooperative networks like Opportunistic Large Arrays (OLAs) [1] are well-suited for multi-hop communication where a group of nodes transmits the same message to another group of nodes. This process continues in a multi-hop manner. With the assumption of single packet transmission and no channel access issues, a necessary condition for correct decoding is that the SNR of received signal is above a certain modulation dependent decoding threshold.

The authors gratefully acknowledge the grant from National ICT R&D Fund, Pakistan, for sponsoring this research work.

Owing to the simplicity in their implementation, OLAs have gained immense popularity. Researchers have mostly studied OLAs using brute force Monte Carlo simulations. The authors in [2] have modeled dense OLA networks under the assumption that the number of nodes goes to infinity with a fixed power per unit area. This model guarantees infinite propagation over a strip network. The continuum assumption, however, renders the model invalid for networks with scarce nodes. In [3], the authors have considered a linear finite OLA network with fixed node placement to analytically model coverage aspects of the network. This work has further been extended in [4] where the absence or presence of a node at a particular location is determined by a Bernoulli point process. Their analysis proves that infinite propagation cannot be guaranteed with finite nodes but the probability of successful delivery at a destination can be calculated. A 2-dimensional grid network is studied with simple assumptions in [5].

The extension of the above models for 2-dimensional ad hoc random networks is not very straight forward. In this paper, we consider a fixed boundary 2-dimensional random network to simplify our design. Many cooperative approaches, specifically for strip-shaped networks also require that the cooperating nodes of two adjacent levels form disjoint sets [3], [6], [7]. We focus on an ad hoc single-input single-output (SISO) strip network where each node is randomly placed in contiguous square areas. This stochastic design will act as a prelude to finding the coverage aspects of fixed boundary cooperative ad hoc networks and for more general strip networks. To find the one-hop success probability, the path loss model in conjunction with the fading model needs to be determined. Since each node is placed randomly, the distance between any two adjacent nodes is also random. While the channel is considered to have Rayleigh fading, the path loss model depends on the distribution of the Euclidean distance between a pair of nodes in contiguous however disjoint areas. To the best of authors' knowledge, the distribution of latter is unknown in literature.

Previous work done in finding the distribution of Euclidean distance focuses on the distance between two nodes in the

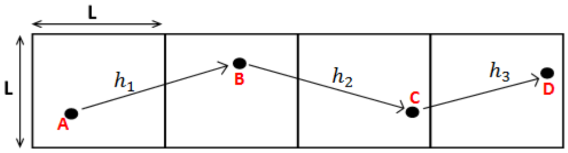

Fig. 1. A realization of SISO network with randomly placed nodes in adjacent levels.

same area, which does not map on to the problem being considered here. Present results cannot be obtained as a mere generalization of the methods given in [8] and the references therein. Many authors derived the distance distribution between two nearest neighbors or n^{th} nearest neighbor as in [9] and [10] but they do not assume disjoint sets of transmitter-receiver regions.

In this paper, we find a closed-form expression for the distribution of the Euclidean distance between two nodes uniformly placed in disjoint but contiguous square regions. This distribution is then approximated to the Weibull distribution by using the moment matching approach to make it tractable. It is further used along with the Rayleigh fading model to compute the closed-form one-hop success and outage probabilities. These probabilities are then used to find the coverage of an ad hoc SISO network performing multi-hop communications.

The organization of the paper is as follows: in Section II, we define the system parameters and introduce some notations. Section III describes the derivation of the distribution of the Euclidean distance and its Weibull approximation. In Section IV, we derive a closed-form expression of the probability law of the power received at a node. Section V provides the results and system performance while Section VI concludes this paper by giving future directions to this study.

II. System Model

Let ϕ denote a simple point process on $\mathcal{M} \in \mathbb{R}^2$ such that $\mathcal{M}$ is a square region of dimension LxL and $\phi(\mathcal{M}) = 1$ with probability one. The random point is uniformly distributed in the bounded set $\mathcal{M}$. This scenario is illustrated in Fig. 1 where the nodes are placed in contiguous, however, disjoint sets. This situation can be attributed to a hallway consisting of adjacent rooms of equal sizes where a sensor node is placed randomly in each room for monitoring purposes. A node transmits to the other node under the impacts of a wireless channel. To model the wireless propagation, the transmitted signal is subject to two independent impairments; a path loss model for a random distance d between the nodes and a small-scale fading model.

The received power, P_r, at a node distance d away from its sender in this SISO network is given as

$$P_r = \frac{P_t h}{d^\beta}, \tag{1}$$

where β is the path loss exponent with a usual range of 2-4, h is a unit-mean exponential random variable (RV), which is the squared envelope of the signal undergoing Rayleigh fading and P_t is the transmit power. Correct decoding of the data is achieved when the received power is greater than a modulation dependent threshold, τ.

To study the probability of coverage of this network, $\mathbb{P}(P_r \geq \tau)$, we need the cumulative distribution function (CDF) of the received power. In Equation. (1), d, is a RV with unknown distribution. Hence, we need to find a generalized probability density function (PDF) of the distance raised to any positive power β. Then the distribution of the power will be the distribution of the ratio of exponential random variable and distance raised to an arbitrary path loss exponent. Multiplication with P_t just scales the ratio distribution.

Throughout this paper, we use the following notations. Upper case alphabets represent RVs and lower case alphabets are their respective realizations.'$F_X(x)$' denotes the CDF of RV X while '$f_X(x)$' denotes its PDF.

III. Distance Distribution

Let $\mathcal{M}_i : i \in \{1, 2\}$ represent adjacent, square regions such that

$$\mathcal{M}_i = \{(x_i, y_i) : 0 \leq x_i \leq L; 0 \leq y_i \leq L\} \quad \forall\, i \in \{1, 2\}. \tag{2}$$

If $A(x_1, y_1)$ and $B(x_2 + L, y_2)$ are two random points, then the Euclidean distance between them is given as

$$d = \sqrt{((x_2 + L) - x_1)^2 + (y_2 - y_1)^2}. \tag{3}$$

In (3), $x_i \in X_i$ and $y_i \in Y_i\ \forall\, i \in \{1, 2\}$, where X_i and Y_i are independent and uniformly distributed (IUD) random variables over $[0, L]$. Without the loss of generality, the x-axis offset can be applied to y-axis and same results can be obtained.

Let $Z \doteq X_2 + L - X_1$. As X_1 and X_2 are both independent uniform random variables, the distribution of Z is obtained using a convolution integral

$$f_Z(z) = \int_{-\infty}^{\infty} f(x - z) f(x - L) dx, \tag{4}$$

where $f_X(x) = 1/L\,[u(x) - u(x - L)]$, where $u(x)$ is the unit step function. Solving the integral we get,

$$f_Z(z) = \begin{cases} 0 & z \leq 0, z > 2L, \\ \frac{z}{L^2} & 0 < z \leq L, \\ \frac{1}{L^2}[2L - z] & L < z \leq 2L. \end{cases} \tag{5}$$

The expression for the PDF, $f_R(r)$, of $R \doteq Y_2 - Y_1$ is given as [10]

$$f_R(r) = \begin{cases} 0 & r \leq -L, r > L, \\ \frac{1}{L^2}[L + r] & -L < r \leq 0, \\ \frac{1}{L^2}[L - r] & 0 < r \leq L. \end{cases} \tag{6}$$

For the next step, we find the PDF, $f_T(t)$, of $T = Z^2$, where $f_T(t)$ is non-zero for $0 \leq t \leq 4L^2$ and is zero elsewhere. Using Theorem 13.2 of [11], a simple derivation yields

$$f_T(t) = \begin{cases} 0 & t \leq 0, t > 4L^2, \\ \frac{1}{L^2} & 0 < t \leq L^2, \\ \frac{1}{L\sqrt{t}} - \frac{1}{2L^2} & L^2 < t \leq 4L^2. \end{cases} \tag{7}$$

Again, we see that because of no offset in the y coordinates, the PDF, $f_Q(q)$, of $Q = R^2$ is given as [10]

$$f_Q(q) = \begin{cases} 0 & q \le 0, q > L^2, \\ \frac{1}{L\sqrt{q}} - \frac{1}{L^2} & 0 < q \le L^2. \end{cases} \quad (8)$$

It is obvious that the distributions of T and Q are not identical. Now, for the expression of the PDF of squared Euclidean distance, we need to find the distribution, $f_U(u)$, of $U = Q + T$, where $d^2 \in U$. Since Q and T are independent, we convolve $f_Q(q)$ and $f_T(t)$ to find $f_U(u)$, which is given as

$$f_U(u) = \begin{cases} 0 & u \le 0, u > 5L^2, \\ \frac{\sqrt{u}}{L^3} - \frac{u}{2L^4} & 0 < u \le L^2, \\ \frac{1}{L^2}\left[\frac{3}{2} + \pi - 2sin^{-1}\sqrt{\frac{L^2}{u}}\right] \\ -\frac{2}{L^3}\left[\sqrt{u} + \sqrt{u - L^2}\right] + \frac{u}{L^4} & L^2 < u \le 2L^2, \\ \frac{1}{L^2}\left[2sin^{-1}\sqrt{\frac{L^2}{u}} - \frac{1}{2}\right] \\ -\frac{2}{L^3}\left[\sqrt{u} - \sqrt{u - L^2}\right] & 2L^2 < u \le 4L^2, \\ \frac{1}{L^2}\left[-\frac{5}{2} + 2tan^{-1}\sqrt{\frac{4L^2}{u-4L^2}} \right. \\ \left. -2tan^{-1}\sqrt{\frac{u-L^2}{L^2}}\right] - \frac{u}{2L^4} + \\ \frac{1}{L^3}\left[2\sqrt{u - L^2} + \sqrt{u - 4L^2}\right] & 4L^2 < u \le 5L^2. \end{cases} \quad (9)$$

It can be noticed from the limits of the above equation that the value of squared distance between two nodes in disjoint areas is now distributed between 0 and $5L^2$.

The PDF $f_V(v)$ of $V = U^{\frac{\beta}{2}}$ is derived by using Theorem 13.2 of [11]. The generalized distribution of distance raised to any power $\beta > 0$ is

$$f_V(v) = \frac{1}{\alpha} v^{\frac{1}{\alpha}-1} f_U(v^{1/\alpha}), \quad (10)$$

where $\alpha = \beta/2$. The factor of 2 is introduced in the exponent of v because we are directly transforming the squared distance distribution.

A. Matching distance distribution with Weibull Distribution

Even though an exact closed-form PDF of the distance is obtained, the expression in (9) consists of various non-linear terms which make further analysis quite tedious. Furthermore, since the PDF satisfies all conditions of regularity [12], a simpler equivalent function can be obtained, which encapsulates the features of the complicated expression of PDF in (9) carrying all information about the parameters L and α. For this reason, we proceed with the moment matching technique such that the first two moments of other random variables with known distributions are matched with the respective moments of U. The expected value of squared distance is calculated as

$$E[U] = \int_0^{5L^2} u f_U(u) du = \frac{4}{3} L^2, \quad (11)$$

where $E[.]$ denotes the expectation operator. Similarly, the second moment of squared distance is given as

$$E[U^2] = \int_0^{5L^2} u^2 f_U(u) du = \frac{227}{90} L^4. \quad (12)$$

It has been noticed that the distribution of squared distance matches closely to the Weibull distribution. Weibull distribution is characterized by its shape and scale parameter (k and λ respectively). A Weibull distributed RV, W, has the mean

$$E[W] = \lambda\Gamma(1 + \frac{1}{k}) \quad (13)$$

and variance

$$Var[W] = \lambda^2\Gamma(1 + \frac{2}{k}) - E[W]^2, \quad (14)$$

where $\Gamma(x)$ is the complete Gamma funtion. As we know $E[W^2] = Var[W] + E[W]^2$, we get the second moment of W as

$$E[W^2] = \lambda^2\Gamma(1 + \frac{2}{k}). \quad (15)$$

Hence, the moment matching approach boils down to solving a set of two non-linear equations simultaneously such that

$$\begin{aligned} \lambda\Gamma(1 + \frac{1}{k}) &= \frac{4}{3} L^2, \\ \lambda^2\Gamma(1 + \frac{2}{k}) &= \frac{227}{90} L^4. \end{aligned} \quad (16)$$

After eliminating λ in (16), the equation is solved numerically for k. We obtain a constant value of k which is equal to 1.5806. The value of λ thus obtained is $4L^2/(3\Gamma(1.6327))$. Hence, the new distribution of the squared distance is now

$$f_U(u) \approxeq \frac{k}{\lambda^k} u^{k-1} e^{-(\frac{u}{\lambda})^k} \qquad u > 0. \quad (17)$$

IV. Outage Probability

To determine the outage probability, the CDF of the received power in (1) needs to be calculated, which is the distribution of the ratio of two RVs namely $H \ni h$ and V. If $P_r \in P$, then we are interested in finding the probability law for P such that $P = P_t H/V$. Using (10), the generalized distribution of the distance raised to any positive power is

$$f_V(v) \approxeq \frac{s}{\chi^s} v^{s-1} e^{-(\frac{v}{\chi})^s} \qquad v > 0. \quad (18)$$

where, $s = k/\alpha$ and $\chi = \lambda^\alpha$. Here we notice that the addition of a path loss exponent simply alters the shape and scale parameters of the Weibull distribution in (17).

Since H is a unit mean exponential RV and V is a Weibull RV, we need to find the distribution of the ratio of an exponential and a Weibull RV. In [13], the authors have derived a closed form expression for the distribution of the ratio of Gamma and Weibull RVs. The PDF of a Gamma random variable, G is

$$f_G(g) = \frac{\theta^\gamma}{\Gamma(\gamma)} g^{\gamma-1} e^{-g\theta} \qquad g > 0. \tag{19}$$

where γ and θ are shape and scale parameters, respectively. We note that when γ and θ are fixed to 1, the Gamma distribution simplifies to

$$f_G(g) = e^{-g} \qquad g > 0, \tag{20}$$

which is essentially the PDF of a unit-mean exponential RV. Hence, $H \equiv G$.

Using (10), the CDF of $P_t G/V$ is given as [13]

$$\begin{aligned} F_P(p) &= \int_0^\infty F_G(pv) f_V(v) dv \\ &= 1 - \frac{s}{\chi^s} \int_0^\infty \Gamma(1, vp) v^{s-1} e^{-\left(\frac{v}{\chi}\right)^s} dv, \end{aligned} \tag{21}$$

where $\Gamma(x, y)$ is the incomplete Gamma function and $P_t = 1$. The integral in (21) is calculated using Equation (2.10.1.5) of [14]. The outage probability, $F_P(\tau/P_t)$, which is $\mathbb{P}(P_r \leq \tau/P_t)$ is given as

for $s < 1$,

$$F_P\left(\frac{\tau}{P_t}\right) = -\left(\frac{\tau\chi}{P_t}\right)^{-s} \sum_{n=0}^{\infty} \frac{\Gamma(1+s+sn)}{(n+1)!} \left(-\frac{P_t^s}{\tau^s \chi^s}\right)^n, \tag{22a}$$

for $s = 1$,

$$F_P\left(\frac{\tau}{P_t}\right) = \frac{\tau\chi}{P_t + \tau\chi}, \tag{22b}$$

and for $s > 1$,

$$F_P\left(\frac{\tau}{P_t}\right) = \frac{\tau\chi}{P_t} \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \Gamma\left(\frac{1+s+n}{s}\right) \left(-\frac{\tau\chi}{P_t}\right)^n. \tag{22c}$$

V. Results and System Performance

In this section, we verify our analytical models and present some useful results pertaining to the performance of SISO network of Fig. 1. Fig. 2 represents the PDF of the squared distance between a pair of nodes for different lengths L. It can be seen that both the numerical results and the analytical results from (9) closely match for all values of L. For the simulation purposes, a pair of nodes is randomly placed in their defined regions and the Euclidean distance between them is computed. The simulation results are averaged over 10000 trials.

The PDF of the squared distance is then plotted and compared with the Weibull PDF in (17) for different values of L as shown in Fig. 3. It can be noticed that the Weibull distribution with computed moments is a close fit to the squared distance

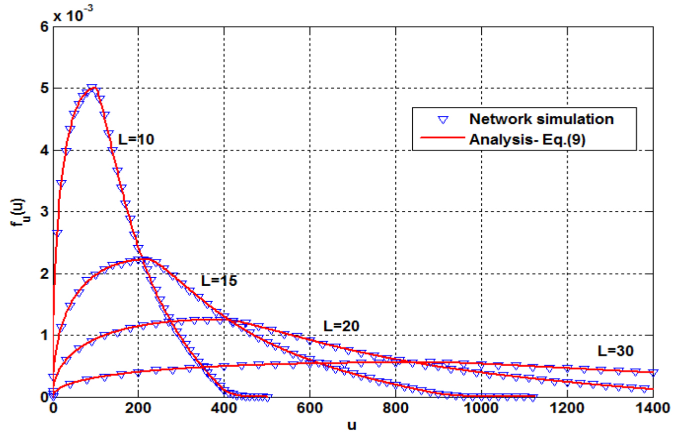

Fig. 2. PDF of the squared distance, $f_U(u)$, for different region lengths L.

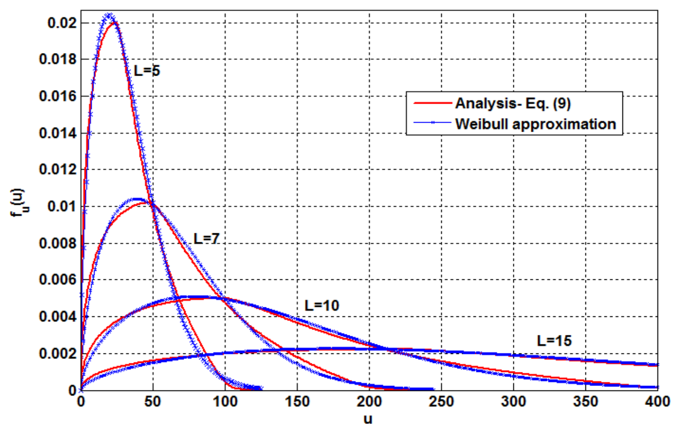

Fig. 3. Comparison of the analytical PDF of the squared distance with the matched Weibull distribution (shape parameter, $k = 1.5806$, scale parameter, $\lambda = 4L^2/(3\Gamma(1.6327))$ for different values of region lengths L.

distribution in (9). When calculating the outage probability of a receiver in presence of fading in wireless systems, we are interested in the complementary CDF (CDFC) or the tail probability. In Fig. 4, we plot the actual CDFC, $F_U^c = 1 - F_U(u)$, calculated from (9) and compare it with the Weibull CDFC with parameters calculated from (16). The mean squared error between the actual and the approximated CDFCs is of the order 10^{-5}. Hence, Weibull distribution is an accurate approximation of the actual distribution.

Figures 5 and 6 display the analytical results generated using (22). It is desired to observe the effects of varying the path loss exponent on the outage probability. Since $s = k/\alpha = 1.5806/\alpha$ and $\alpha = \beta/2$, this implies that $s = 3.1612/\beta$, where β is the path loss exponent. When $\beta = 2$, we use Eq. (22c) to plot the coverage probability vs. P_t in Fig. 5 for $\tau = 0.001$ and $L = 10$. Similarly, we use Eq. (22b) and Eq. (22a) for $\beta = 3.1612$ and $\beta = 4$, respectively. It can be noticed that as the path-loss exponent increases, the outage probability at an arbitrary transmit power increases.

In multi-hop SISO networks, we are also interested in

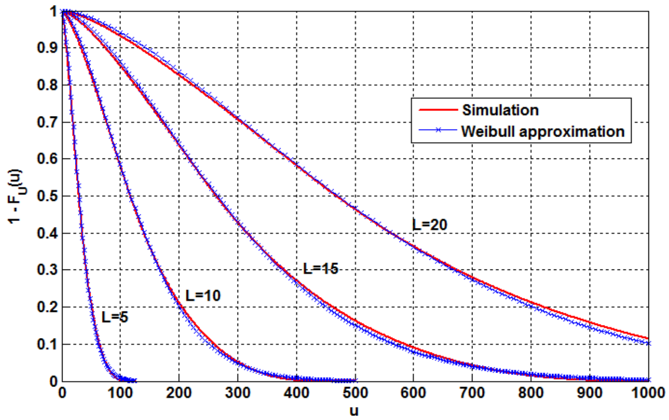

Fig. 4. Tail probabilities of analytical model and the Weibull approximation for different values of L.

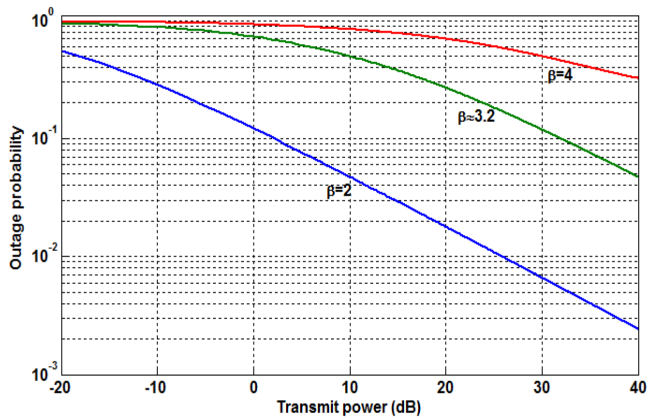

Fig. 5. Effects of transmit power on the outage probability for different path loss exponents β. $L = 10$, $\tau = 0.001$.

finding the coverage of the network. The one-hop success probability is given as $\rho = 1 - F_P(\tau/P_t)$. If a message propagates m hops through the strip, the m-hop success probability will be ρ^m. If it is desired that the m-hop success probability should be greater than or equal to η, where η is the quality of service (QoS), then $\rho^m \geq \eta$, which implies

$$m \leq \frac{\ln \eta}{\ln \rho}. \tag{23}$$

The average coverage range (CR) is, therefore, given as

$$CR = mL. \tag{24}$$

The CR is plotted for different values of transmit power for a fixed β and threshold in Fig. 6. It can be observed that for the same transmit power, the message propagates to a larger distance when the width of the network is small. This behavior can be attributed to the fact that path loss plays a significant role in reducing the coverage of a network. As we increase L, we are effectively increasing the signal attenuation because the average distance between a pair of node increases.

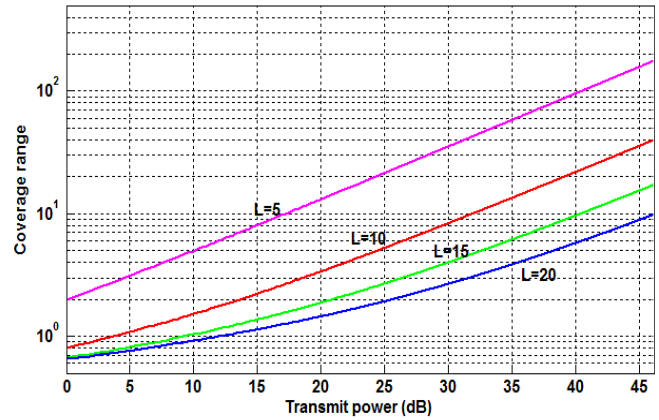

Fig. 6. Effects of transmit power on coverage range for different region lengths L. $\beta \approx 3.2$, $QoS = 0.9$, $\tau = 0.001$.

VI. Conclusion and Future Work

In this paper, we have derived the exact PDF of the squared distance between a pair of nodes placed randomly in contiguous but disjoint areas in a strip network. This distribution is generalized for all positive powers of distance. In order to make this distribution function manageable for further analysis, we approximated it with the well-known Weibull PDF. This approximation is done by computing the first two moments of the squared distance and then equating them to the respective moments of the Weibull distribution. The approximated Weibull PDF is then used to find the outage probability of a node in presence of Rayleigh fading. We have presented the CDF of the ratio of an exponential and a Weibull RV, which is valid for all positive real values of the shape parameter, s. To the best of our knowledge, this ratio distribution for all positive real values of s is not explicitly present in the relevant literature. As an extension to our work, it is recommended to introduce transmit diversity in this network whereby, multiple nodes will be transmitting the same message to a receiver with uncorrelated channel gains.This would require the computation of the distribution of the sum of received power. The calculated one-hop success probabilities would be useful in studying the coverage aspects of more realistic cooperative strip networks.

References

[1] A. Scaglione and Y.-W. Hong, "Opportunistic large arrays: Cooperative transmission in wireless multihop ad hoc networks to reach far distances," *IEEE Trans. Signal Processing,*, vol. 51, no. 8, pp. 2082–2092, 2003.

[2] B. Sirkeci-Mersen and A. Scaglione, "A continuum approach to dense wireless networks with cooperation," in *Proc. IEEE INFOCOM*, vol. 4, pp. 2755–2763, IEEE, Mar. 2005.

[3] S. A. Hassan and M. A. Ingram, "A quasi-stationary markov chain model of a cooperative multi-hop linear network," *IEEE Trans. Wireless Commun.*, vol. 10, pp. 2306–2315, July 2011.

[4] S. A. Hassan and M. A. Ingram, "On the modeling of randomized distributed cooperation for linear multi-hop networks," in *Proc. IEEE Intl. Conf. Commun. (ICC)*, pp. 366–370, IEEE, 2012.

[5] S. A. Hassan, "Performance analysis of cooperative multi-hop strip networks," *Wireless Personal Communications*, pp. 1–10, June 2013.

[6] L. V. Thanayankizil and M. A. Ingram, “Opportunistic large array concentric routing algorithm (OLACRA) over wireless fading channels,” in *IEEE Globecom Workshops*, pp. 1–5, 2007.

[7] Y. J. Chang, H. Jung, and M. A. Ingram, “Demonstration of an OLA-based cooperative routing protocol in an indoor environment,” in *Proc. 17th European Wireless Conf.*, pp. 1–8, VDE, Apr. 2011.

[8] D. Moltchanov, “Survey paper: Distance distributions in random networks,” *Ad Hoc Netw.*, vol. 10, pp. 1146–1166, Aug. 2012.

[9] S. Srinivasa and M. Haenggi, “Distance distributions in finite uniformly random networks: Theory and applications,” *IEEE Trans. Veh. Technol.*, vol. 59, no. 2, pp. 940–949, 2010.

[10] A. Kostin, “Probability distribution of distance between pairs of nearest stations in wireless network,” *Electronics Letters*, vol. 46, no. 18, pp. 1299–1300, 2010.

[11] S. A. Klugman, H. H. Panjer, G. E. Willmot, and G. G. Venter, *Loss Models: From Data to Decisions*, vol. 2. Wiley New York, 1998.

[12] S. M. Kay, *Fundamentals of Statistical Signal Processing, volume I: Estimation Theory (v. 1).* Prentice Hall, 1998.

[13] S. Nadarajah and S. Kotz, “On the product and ratio of Gamma and Weibull random variables,” *Econometric Theory*, vol. 22, no. 02, pp. 338–344, 2006.

[14] A. Prudnikov, Y. A. Brychkov, and O. Marichev, *Integrals and Series: Special Functions, vol. II.* Gordon and Breach Science Publishers, 1986.