Imperfect Timing Synchronization in Multi-Hop Cooperative Networks: Statistical Modeling and Performance Analysis

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***Abstract*—This paper studies a multi-hop cooperative network operating under imperfect timing synchronization. The timing errors at the receiving nodes are statistically modeled and the effects on the signal-to-noise ratio (SNR) are derived. The distribution of SNR is then used to find the outage probability of the nodes in a hop which together with a Markov model provides the probability of making a successful hop. The results show that the timing synchronization errors reduce the one-hop success probability and the coverage of the network.**

I. Introduction

Cooperative transmission (CT) introduced a new dimension of data propagation in a more reliable manner with enhanced system capacity. The benefits of using CT in sensor networks have been shown in a variety of performance metrics such as range extension and energy-efficiency [1]. Although CT provides diversity and array gains in wireless networks, however, a major issue of CT lies in the timing synchronization. To achieve maximum diversity gain, the coherent combining techniques require a perfect alignment of all replicas of the received signal. This condition, however, doesn't hold in general and each copy of signal experiences a distinct timing offsets due to its propagation delay to the receiver. Most of the previous works, for instance [1], assume perfect timing synchronization in these networks.

A considerable amount of work has been done in timing synchronization of cooperative networks. The authors in [2] suggested a transmit time synchronization (TTS) method to synchronize the transmissions of intra-cluster transmissions. On the other hand, receiver timing synchronization (RTS) is proposed in [3] and [4]. However, the network is limited to two-hop and no insight into multi-hop propagation is considered.

In this paper, we consider a decode and forward multi-hop cooperative network using TTS of [2] and RTS of [3] at each hop and quantitatively evaluate the performance of the network subject imperfect timing synchronization. Specifically, we derive a statistical model for timing synchronization errors at a hop and use it to calculate the probability of making a successful hop. We compare our results with the perfectly synchronized case and quantify the signal-to-noise ratio (SNR) margin that is required to get the same coverage of the multi-hop network under consideration.

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II. System Model

Consider a multi-hop network topology given in [2]. We assume that the nodes are partitioned into non-overlapping levels or hops and each node in a hop is d distance apart from its adjacent transmitting node. The distance between two consecutive hops is Md, where M is the number of nodes in each level. The node which successfully decodes the received signal can take part in further transmissions and is called decode-and-forward (DF) node. The set of indices of DF nodes at level n is represented by $\mathbb{N}_n$. We assume a quasi-static and flat fading Rayleigh channel.

III. Statistical Analysis of Timing Offset

In this section, we present the statistics of the estimated timing offset, which will be used subsequently for finding the success probability of a node. Let's denote the timing offset corresponding to the signal transmitted by the m^{th} node in level $n-1$ to the k^{th} node in level n by $\tau_{m,k}^{(n)}$ and the estimated timing offset is denoted by $\hat{\tau}_{m,k}^{(n)}$. At the k^{th} node of the level n, all the estimated offsets, $\hat{\tau}_{\ell,k}^{(n)} \forall \ell = \{1, 2, \ldots, M\}$ are combined to get a single optimal sampling time $\hat{\tau}_k^{(n)}$ using [3, Eq.(30)]. This $\hat{\tau}_k^{(n)}$ is used to pre-synchronize the clock of k^{th} node. The mean of $\hat{\tau}_k^{(n)}$ is given as $\mu_{\hat{\tau}_k^{(n)}} = \sum_{k=1}^{M} \zeta_{m,k}^{(n)} \tau_{m,k}^{(n)}$, where

$$\zeta_{m,k}^{(n)} = \sum_{\substack{\ell=1 \\ \ell \neq m}}^{M} \left[B_{\ell,k}^{(n)} \mu_{m,k}^{(n)} \frac{\mu_{\ell,k}^{(n)} \left(\ln(\mu_{\ell,k}^{(n)}/\mu_{m,k}^{(n)}) - 1 \right) + \mu_{m,k}^{(n)}}{\left(\mu_{\ell,k}^{(n)} - \mu_{m,k}^{(n)} \right)^2} \right],$$

where $B_{\ell,k}^{(n)} = \prod_{\substack{j=1 \\ j \neq \ell \\ j \neq m}}^{M} \frac{\mu_{\ell,k}^{(n)}}{\mu_{\ell,k}^{(n)} - \mu_{j,k}^{(n)}}$, $\qquad \mu_{m,k}^{(n)} = 1/(\xi_{m,k})^{\beta}$.

The $\xi_{m,k}$ is the Euclidean distance between m^{th} node of level $n-1$ to k^{th} node of level n and β is path-loss exponent. The variance of $\hat{\tau}_k^{(n)}$ is given as

$$\begin{aligned} \sigma^2_{\hat{\tau}_k^{(n)}} &= \sum_{m=1}^{M} \left[\theta_{m,k}^{(n)} \left((\sigma_{m,k}^{(n)})^2 + \left(\tau_{m,k}^{(n)} \right)^2 \right) \right] \\ &+ \sum_{p=1}^{M} \sum_{\substack{q=1 \\ q \neq p}}^{M} \left[\left(\zeta_{q,k}^{(n)} \zeta_{p,k}^{(n)} + \delta_{k_{q,p}}^{(n)} \right) \tau_{p,k}^{(n)} \tau_{q,k}^{(n)} \right] - \mu^2_{\hat{\tau}_k^{(n)}}, \end{aligned} \tag{1}$$

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where $\theta_{m,k}^{(n)}$ is given as

$$\theta_{m,k}^{(n)}=\sum_{\substack{\ell=1\\ \ell\neq m}}^{M}\left[B_{\ell,k}^{(n)}\mu_{m,k}^{(n)}\frac{\left(\mu_{\ell,k}^{(n)}\right)^2-\left(\mu_{m,k}^{(n)}\right)^2-2\mu_{\ell,k}^{(n)}\mu_{m,k}^{(n)}\ln(\mu_{\ell,k}^{(n)}/\mu_{m,k}^{(n)})}{\left(\mu_{\ell,k}^{(n)}-\mu_{m,k}^{(n)}\right)^3}\right] \tag{2}$$

The $\delta_{k_{q,p}}^{(n)}$ is the covariance between $\omega_{p,k}^{(n)}$ and $\omega_{q,k}^{(n)}$, where $\omega_{\ell,k}^{(n)} = |\alpha_{\ell,k}^{(n)}|^2/\sum_{m\in\mathbb{N}_{n-1}}|\alpha_{m,k}^{(n)}|^2$. The $|\alpha_{\ell,k}^{(n)}|^2$ denotes the channel gain from m^{th} node of level $n-1$ to k^{th} node of level n and $|\alpha_{\ell,k}^{(n)}|^2\sim\ Exp((\xi_{m,k})^{\beta})$. The $(\sigma_{m,k}^{(n)})^2 \approx c(L_o,M)\sigma_N^2(\xi_{m,k}^{(n)})^{\beta}/P_t$ is the variance of estimator, where $c(L_o,M)$ is a constant related to the length of the training sequence, L_o, and the number of relays per cluster, M [3]. The P_t is the transmit power of each node and σ_N^2 denotes the variance of the additive white Gaussian noise (AWGN).

IV. SNR in Presence of Timing Errors

Consider a perfect channel state information (CSI) and that the replicas of received signal at each node are combined using pre-detection maximal ratio-combining (MRC). Assuming a rectangular pulse shaping, the SNR is not affected due to imperfect timing synchronization if two consecutive bits have the same phase. In this case, the SNR of the received signal at k^{th} node at n^{th} level is given as $\gamma_k^{(n)}=\frac{P_t}{\sigma_N^2}\sum_{m\in\mathbb{N}_{n-1}}\left|\alpha_{m,k}^{(n)}\right|^2$. It should be noted that $\gamma_k^{(n)}$ is also equal to the SNR with perfect synchronization case. The SNR when two adjacent bits have opposite phase, i.e., when they are antipodal, is reduced and is given as

$$\tilde{\gamma}_k^{(n)}=\frac{P_t}{\sigma_N^2}\frac{\left[\sum_{m\in\mathbb{N}_{n-1}}\left(\left|\alpha_{m,k}^{(n)}\right|^2\left(1-2\left|\tau_{m,k}^{(n)}-\hat{\tau}_k^{(n)}\right|\right)\right)\right]^2}{\sum_{m\in\mathbb{N}_{n-1}}\left|\alpha_{m,k}^{(n)}\right|^2}. \tag{3}$$

It should be noticed that $\tilde{\gamma}_k^{(n)}$ reduces to $\gamma_k^{(n)}$ if the estimated timing offset $\hat{\tau}_k^{(n)}$ is perfect. However, the error magnifies as the difference between actual and estimated timing offset increases. To find the success probability of the k^{th} node, $\mathbb{P}\{\tilde{\gamma}_k^{(n)}>\gamma_{th}\}$ is required, which involves the computation of the probability density function (PDF) of $\tilde{\gamma}_k^{(n)}$, which is prohibitive in closed-form. However, an upper-bound on $\tilde{\gamma}_k^{(n)}$ can be found by using the Cauchy-Schwarz inequality. Thus using this inequality, we have

$$\tilde{\gamma}_k^{(n)}\leq\frac{P_t}{\sigma_N^2}\sum_{m\in\mathbb{N}_{n-1}}\left[\left|\alpha_{m,k}^{(n)}\right|^2\left(1-2\left|\tau_{m,k}^{(n)}-\hat{\tau}_k^{(n)}\right|\right)^2\right]. \tag{4}$$

The success probability for each node in a level is evaluated in the subsequent section where a Markov model of transmission process is formulated.

V. Model of the Network

We represent the state of the k^{th} node at level n by a binary indicator function $\mathbb{I}_k(n)$, where $\mathbb{I}_k(n)=1$ represents the successful decoding while $\mathbb{I}_k(n)=0$ represents a failure in decoding. Thus, the state of level n is represented by $\mathcal{X}(n)=[\mathbb{I}_1(n),\mathbb{I}_2(n),\dots,\mathbb{I}_M(n)]$. The state of level n depends only upon the state of level $n-1$, thereby, making $\mathcal{X}$ a memoryless Markov chain with an absorbing state and $\mathcal{S}$ transient states. If the transitions to and from the absorbing state are eliminated, the resulting transition probability matrix, $\mathbf{P}$ is a right sub-stochastic and irreducible having a dimension of $(2^M-1)\times(2^M-1)$. Considering these properties of $\mathbf{P}$, the Perron-Frobenius theorem is exploited [1], which states that there exists a maximum eigenvalue ρ and an associated left eigenvector $\mathbf{u}$, such that $\mathbf{uP}=\rho\mathbf{u}$. In our case, ρ provides the one-hop success probability, which implies that at least one node in the next level will decode the message.

A. Transition Probability Matrix

Assuming that two adjacent bits can have same or opposite phase with equal probability, the conditional probability of successful decoding of data at the k^{th} node at level n is given as

$$\mathbb{P}\{\mathbb{I}_k(n)=1|\psi\}=\frac{1}{2}\mathbb{P}\{\gamma_k^{(n)}>\gamma_{th}|\psi\}+\frac{1}{2}\mathbb{P}\{\tilde{\gamma}_k^{(n)}>\gamma_{th}|\psi\}, \tag{5}$$

where $\psi=\{\mathcal{X}(n-1)\in\mathcal{S}\}$ is the state of level $n-1$. The first term, $\mathbb{P}\{\gamma_k^{(n)}>\gamma_{th}|\psi\}$ gives the probability of success at the k^{th} node at level n given two adjacent bits have the same phase, while the second term, $\mathbb{P}\{\tilde{\gamma}_k^{(n)}>\gamma_{th}|\psi\}$ gives the success probability if two adjacent bit are antipodal. These two terms are combined by using the law of total probability. The γ_{th} is the modulation dependent SNR threshold for successful decoding of received signal. The first term of (5) can be evaluated by using the cumulative density function (CDF) of hypoexponential distribution [1], which is given as

$$\mathbb{P}\{\gamma_k^{(n)}>\gamma_{th}|\psi\}=\sum_{m\in\mathbb{N}_n^{(i)}}C_m^{(k)}\exp\left(-\lambda_m^{(k)}\gamma_{th}\right), \tag{6}$$

where $C_m^{(k)}=\prod_{\substack{\ell\in\mathbb{N}_{n-1}^{(i)}\\ \ell\neq m}}\frac{\lambda_\ell^{(k)}}{\lambda_\ell^{(k)}-\lambda_m^{(k)}}$, $\lambda_m^{(k)}=\frac{(\xi_{m,k}(n))^{\beta}\sigma_N^2}{P_t}$, and $\mathbb{N}_{n-1}^{(i)}$ is the set of DF nodes at level $n-1$ which are 1 contributing to state i. For the second term of (5), the PDF of SNR given in (3) is prohibitive in closed-form. However, by using CDF of upper-bound given in (4), the second term of (5) is evaluated as

$$\mathbb{P}\{\tilde{\gamma}_k^{(n)}>\gamma_{th}|\psi\}=\sum_{m\in\mathbb{N}_{n-1}^{(i)}}\int_{-\infty}^{\infty}D_m^{(k)}(x)\exp\left(-\tilde{\lambda}_m^{(k)}(x)\,\gamma_{th}\right)f_{\hat{\tau}_k^{(n)}}(x)dx, \tag{7}$$

where $D_m^{(k)}(x)=\prod_{\substack{\ell\in\mathbb{N}_{n-1}^{(i)}\\ \ell\neq m}}\frac{\tilde{\lambda}_\ell^{(k)}(x)}{\tilde{\lambda}_\ell^{(k)}(x)-\tilde{\lambda}_m^{(k)}(x)}$, $\tilde{\lambda}_m^{(k)}(x)=\frac{(\xi_{m,k}^{(n)})^{\beta}\sigma_N^2}{P_t\left(1-2\left|\tau_{m,k}^{(n)}-x\right|\right)^2}$.

In (7), $f_{\hat{\tau}_k^{(n)}}(x)$ is the PDF of the $\hat{\tau}_k^{(n)}$. The integration (7) is very difficult to solve analytically, however, we can numerically evaluate it by using the Gauss-Hermite quadrature integration. Considering $\Phi^{(k)}=\mathbb{P}\{\mathbb{I}_k(n)=1|\psi\}$ and $\mathbb{N}_n^{(j)}$ and $\overline{\mathbb{N}}_n^{(j)}$ be the indices of nodes in the level n which are 1 and 0, respectively, corresponding to state j, we have following one-step transition probability from i^{th} state to j^{th} state

$$\mathbb{P}_{ij}=\prod_{k\in\mathbb{N}_n^{(j)}}\Phi^{(k)}\prod_{k\in\overline{\mathbb{N}}_n^{(j)}}\left(1-\Phi^{(k)}\right). \tag{8}$$

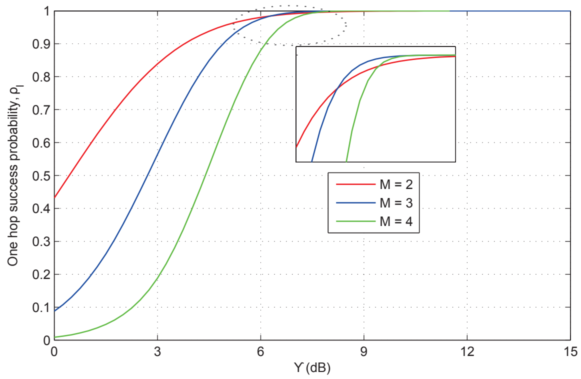

Fig. 1: The one-hop probability of success versus SNR and for different values of M; $d = 1m$, $\gamma_{th} = -5$dB, $L_o = 192$ and $\beta = 2$.

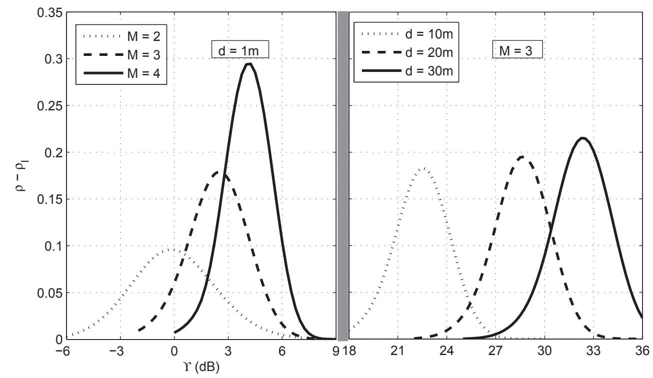

Fig. 2: $\rho - \rho_I$ versus SNR for different values of M and d; $\gamma_{th} = -5$dB, $L_o = 192$ and $\beta = 2$.

Similarly, the one-step transition probability for perfect synchronization case can be found by replacing $\Phi^{(k)}$ in (8) by $\Psi^{(k)} = \mathbb{P}\{\gamma_k^{(n)} > \gamma_{th}|\psi\}$.

VI. Results and Performance Analysis

In this section, we present the analytical results corresponding to various network configurations. We denote the one-hop success probability for the perfectly synchronized case as ρ, while ρ_I is used for the imperfect case and $\Upsilon = P_t/\sigma_N^2$.The analytical results are obtained by finding $\mathbf{P}$ using (21) and plotting its highest eigenvalue.

Fig. 1 shows the behavior of ρ_I versus SNR for various values of number of nodes per hop, M. It can be seen that ρ increases as we increase the SNR. However, at low values of SNR, the $M = 2$ case performs better than the rest of the cases because not only the path-loss and path length disparities are small but also the average distance to the next level nodes is less. However, as the SNR increases the other two cases also perform well. It can be seen in the inset that at very high SNR, the diversity gains start to play their role and $M = 4$ case outperforms the other two cases.

In Fig. 2, $\rho - \rho_I$ is depicted for varying number of nodes per cluster, M and distance d. It can be observed that the difference between two cases is dominant at specific values of SNR for specific value of M. The difference increases as the M increases. At very high SNR, the difference in the success probabilities reduces to zero, which indicates that at high SNR, the effects of timing errors vanish. It is also interesting to note that the spread of these bell-shaped curves decreases as M increases. This is because of the diversity gain at higher M that causes the ρ_I to converge to ρ for even a slight increase in the SNR. The degradation is the success probability due to imperfect synchronization is compensated by applying an additional SNR margin. For instance, at 5.2dB SNR for $M = 4$ case, the timing errors reduces the one-hop success probability to 20%, which should be compensated by applying a suitable SNR margin of 2.8 dB to get the same success probability as that of perfect synchronized case. It can also be noticed in Fig. 2 that by increasing the distance d the peak value of $\rho - \rho_I$ increases due to increased path delay disparities. For instance, if we compare the peaks of the bell-shaped graphs for $d = 10m$ and $d = 30m$, we can notice a peak difference of 4% in one-hop probability of success. Additionally, a higher SNR margin is required due to greater path-loss.

For a given minimum required one-hop success probability η, the coverage for both the perfect and imperfect timing synchronization schemes is given in TABLE I. It can be observed that there is a huge difference between the both perfect and imperfect synchronization cases specially at $M = 4$. We can see a 60% and 56% reduction in coverage for $M = 3$ and $M = 4$, respectively, due to imperfect synchronization. However, the coverage for $M = 2$ is not changed appreciably because it incurs less path disparities.

TABLE I: The coverage of network; $\eta = 99.9\%$, $d = 2m$, $\Upsilon = 9dB$, $\gamma_{th} = -5$dB, $L_o = 192$ and $\beta = 2$.

Synchronization	Coverage distance (m)		
	$M = 2$	$M = 3$	$M = 4$
Perfect	4	56	110
Imperfect	2	22	48

VII. Conclusion

In this paper, we quantified the consequences of imperfect timing synchronization on the performance of cooperative multi-hop networks. The results indicated that even in the presence of transmit and receive synchronization algorithms, the system suffers with reduced SNR and hence the coverage of the network is reduced.

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