On the Estimation of Modulation Index for Binary Full Response CPM Signals

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***Abstract*— This letter considers a full response binary continuous phase modulation (CPM) scheme with a single modulation index. Two estimators are proposed for estimating the modulation index of the underlying CPM scheme. The first estimator is based on the method of moments (MoM) that makes use of the auto-correlation function of the received signal whereas the second estimator is the best linear unbiased estimator (BLUE), which is based on an approximate linear model. A closed-form expression of the modified Cramer-Rao bound (MCRB) is also derived. The MoM estimator is developed only for the rectangular phase shaping functions, while the MCRB expression and the BLUE estimator are applicable to signals with arbitrary phase shaping functions. Numerical performance analysis in terms of mean-squared error and bit-error rate shows that the method of moments estimator performs better at low signal-to-noise ratio (SNR), whereas the performance of the BLUE is close to MCRB at high SNR.**

***Index Terms*— Estimation, continuous phase modulation, Cramer Rao bound, BER, SNR.**

I. Introduction

THE constant envelope nature and bandwidth efficiency are key characteristics that make CPM attractive for several applications [1], [2]. One of the most important parameters of CPM is its modulation index as it permits a trade-off between bandwidth and performance. The precise knowledge of modulation index at receiver is critical for satisfactory performance. This knowledge may not be available to the receiver owing to the transmitter being non-cooperative or being unable to precisely control the oscillator gain, the latter being true for cost effective analog designs.

The problem of modulation index estimation has been undertaken earlier, such as [2], [3] (data-aided) and [4], [5] (non data-aided). The estimator in [2] is based upon the maximum likelihood (ML) criterion that selects the best element among a finite set of possible modulation indices. The DA-LNDIF estimator [3] is based upon Laurent decomposition and a quasi-ML criterion, and it performs very well at low SNR. However, it is not efficient, i.e., its variance does not approach the MCRB at high SNR. A closed-form non data-aided estimator is proposed in [4], which is based upon higher-order statistics (HOS), specifically, a ratio of fourth-order cumulants. A comprehensive treatment of the HOS of the CPM signal using fraction-of-time probability framework has been made in [6]. More recently, cyclic second-order statistics of one-sided CPM signal have been derived using Laurent decomposition [7]. The HOS estimator exhibits reasonable performance at low signal-to-noise ratio (SNR) but its performance does not improve considerably at high SNR due to slower convergence of fourth-order cumulants. The work in [5] used cyclo-stationary properties of the CPM signal and proposed a numerical non data-aided iterative algorithm (NDA-CYC). However, the performance of NDA-CYC estimator is unacceptable at low SNR due to the phase unwrapping errors [3].

Apart from employing estimators, other approaches proposed to deal with the modulation index mismatch problem include the development of robust CPM waveforms [8] and robust receiver design without explicit modulation index estimation [9]. The robust MAP receiver [9] is an iterative soft output algorithm and is based upon the property of the principal Laurent pulse being almost insensitive to the change in modulation index. Although the receiver is iterative and requires prior distribution of the parameter, but it is considerably robust even with modest mismatch. The novel CPM waveforms proposed in [8] use binary to ternary pre-coders to limit the phase evolution and achieve robust performance at the expense of slight increase in receiver complexity and loss in spectral efficiency.

In this letter, we propose two new estimators and derive the MCRB for modulation index [10]. The MCRB is also a lower bound on variance of parameter estimators (like Cramer Rao lower bound [11, p. 27]) and is useful in the presence of nuisance parameters. The first proposed estimator is a non data-aided estimator using the MoM approach and is developed here only for the binary full response CPM signals with rectangular phase shaping function. A comparison of the MSE reveals that the proposed estimator outperforms both non data-aided estimators, i.e., HOS [4] and NDA-CYC [5] at low SNRs. The second proposed estimator is the data-aided BLUE applied to the differential phase of the received signal. In order to demonstrate that MSE performance of an estimator also contributes to the overall performance of the receiver, we perform a BER performance comparison between the proposed BLUE algorithm among data-aided estimators, i.e., ML [2] and DA-LNDIF [3]. The results demonstrate that BLUE has better performance at high SNR and that its MSE gets close to the MCRB. We also observe that even for the robust modulation waveforms [8], when there is mismatch situation, using BLUE estimator prior to demodulation gives considerable performance gain.

II. CPM Signal Description

The complex envelope of a CPM signal is expressed as

$$s(t) = e^{j\phi(t,\alpha)}, \tag{1}$$

Manuscript received February 5, 2018; accepted February 14, 2018. Date of publication March 6, 2018; date of current version May 8, 2018. The associate editor coordinating the review of this letter and approving it for publication was D. Darsena. *(Corresponding author: Syed Asad Ullah.)*

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Digital Object Identifier 10.1109/LCOMM.2018.2812793

and the excess phase $\phi(t,\alpha)$ is defined as

$$\phi(t,\alpha) = 2\pi h \sum_{i=0}^{m} \alpha_i q(t - iT), \quad \text{for } t \leq mT, \tag{2}$$

where the data symbols $\alpha_i \in \{\pm 1\}$ are independent and uniformly distributed and are components of the data sequence α. Here, T is the symbol duration, h is the modulation index, and $q(t)$ is the phase shaping function. The derivative of $q(t)$ is called the frequency shaping function and is denoted by $g(t)$. The function $g(t)$ has a support of L symbol intervals and an underlying area of 1/2. A CPM signal with $L = 1$ is known as full response. In this work, we focus only on binary full-response modulation schemes. The samples of the CPM signal in (1) are defined as

$$s[n] := s(t)|_{t=nT_s} = e^{j\phi[n,\alpha]}, \tag{3}$$

where T_s is the sampling interval and $\phi[n,\alpha] := \phi(t,\alpha)|_{t=nT_s}$ are the samples of the excess phase. The samples are considered for $n \in \{0, 1, \ldots, NN_s - 1\}$, where $N_s := T/T_s$ is the number of samples per symbol and N is the number of observed symbols. The samples of the excess phase are represented as

$$\phi[n,\alpha] = 2\pi h \sum_{i=0}^{m} \alpha_i q_{n-iN_s}, \tag{4}$$

where $m = \lfloor \frac{n}{N_s} \rfloor$ is the symbol index and $\lfloor . \rfloor$ represents the floor operation. The received signal $r[n] = s[n] + w[n]$ is observed in additive white Gaussian noise (AWGN) of variance σ^2. Here, we assume that an appropriate filter has been applied prior to sampling at the receiver to limit the variance of the noise.

III. Estimators

In this letter, we propose two estimators. The first estimator is based on the method of moments [11, p. 289] and the second estimator uses an approximate linear model [12].

A. MoM Estimator

The proposed MoM estimator is applicable only to the rectangular phase shaping functions. The auto-correlation function for binary full response CPM with rectangular phase shaping function for $0 < \tau \leq T$ is defined from [1, p. 361] as

$$R(\tau) = \left(1 - \frac{\tau}{2T}\right)\cos\left(\frac{\pi h \tau}{T}\right) + \frac{1}{2\pi h}\sin\left(\frac{\pi h \tau}{T}\right). \tag{5}$$

Now evaluating (5) at $\tau = T/2$ and $\tau = T$, we get

$$\gamma_1 = \frac{3}{4}\cos\left(\frac{\pi h}{2}\right) + \frac{\sin\left(\frac{\pi h}{2}\right)}{2\pi h} \tag{6}$$

and

$$\gamma_2 = \frac{1}{2}\cos(\pi h) + \frac{\sin(\pi h)}{2\pi h}, \tag{7}$$

respectively, We used both equations for a single unknown (i.e., h), because a single equation doesn't give us a closed-form relationship of h. Combining (6) and (7) and defining $y = \cos(\frac{\pi h}{2})$, we get

$$y^2 - 4\gamma_1 y + 2\gamma_2 + 1 = 0. \tag{8}$$

The solution of the above quadratic equation is given as

$$y = 2\gamma_1 - \sqrt{4\gamma_1^2 - 2\gamma_2 - 1}.$$

for $h \in [0, 1]$. The MoM estimator can thus be expressed as

$$\hat{h}_{MoM} = \frac{2}{\pi}\cos^{-1}\left(2\hat{\gamma}_1 - \sqrt{4\hat{\gamma_1^2} - 2\hat{\gamma}_2 - 1}\right), \tag{9}$$

where $\hat{\gamma}_1$ and $\hat{\gamma}_2$ are the consistent estimators for γ_1 and γ_2, respectively. Although, the validity of the closed-form MoM estimator in (9) is limited to the waveform with rectangular phase shaping function, a similar derivation can be performed for other waveforms. However, it must be mentioned that the close-form expression may not result for other phase shaping functions and a numerical approach may be required.

B. Linear Estimator

The *estimator proposed* in this section is inspired from the phase-based single frequency estimation [12]. The estimator is based on an approximate linear model of the phase of the received signal. According to this model, the received signal $r(n)$, at high SNR, can be approximated as

$$r[n] \approx e^{j(\phi[n,\alpha]+u[n])}, \quad \text{for } n = 0, 1, \ldots, NN_s - 1, \tag{10}$$

where $u[n]$ is modeled as zero mean white Gaussian noise with variance $\sigma^2/2$. Thus the phase of received signal, considering the above approximation, is expressed as

$$\angle r[n] = 2\pi h \sum_{i=0}^{m} \alpha_i q_{n-iNs} + u[n]. \tag{11}$$

To avoid phase unwrapping, the estimator is derived by using the phase differences, i.e.,

$$\Delta_n = \angle r[n+1] - \angle r[n], \quad \text{for } n = 0, 1, \ldots, NN_s - 2. \tag{12}$$

Using (11), the phase difference is given as

$$\Delta_n = 2\pi h \alpha_m [q_{[n+1]_{N_s}} - q_{[n]_{N_s}}] + u[n+1] - u[n], \tag{13}$$

where $[.]_{N_s}$ represent the N_s modulo operator. The vector form of (13) becomes

$$\boldsymbol{\Delta} = \boldsymbol{\omega} h + \boldsymbol{\Omega}, \tag{14}$$

where $\boldsymbol{\Delta} = [\Delta_0, \Delta_1, \ldots, \Delta_{NN_s-2}]^T$, and similarly, $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ are the $(NN_s - 1) \times 1$ column vectors with n-th element defined as $\omega_n = 2\pi\alpha_m[q_{[n+1]_{N_s}} - q_{[n]_{N_s}}]$ and $\boldsymbol{\Omega}_n = u[n+1] - u[n]$, respectively. The noise vector $\boldsymbol{\Omega}$, is Gaussian with zero mean and covariance matrix $\mathbf{C}_\Omega$ as given in [12]. The best linear unbiased estimator (BLUE) [11, p. 141] for h is given by

$$\hat{h}_{BLUE} = \frac{\boldsymbol{\omega}^T \mathbf{C}_\Omega^{-1} \boldsymbol{\Delta}}{\boldsymbol{\omega}^T \mathbf{C}_\Omega^{-1} \boldsymbol{\omega}} \tag{15}$$

and the variance of the estimator $\hat{h}_{BLUE}$ at high SNR will be approximately given by

$$Var(\hat{h}_{BLUE}) = \frac{1}{\boldsymbol{\omega}^T \mathbf{C}_\Omega^{-1} \boldsymbol{\omega}}. \tag{16}$$

IV. MCRB

In this section, we derive the MCRB for the problem of estimation of modulation index from the samples of the binary full response CPM signal in AWGN. The transmitted signal can be represented as $s[n] = \mu_n + j\upsilon_n$ where $\mu_n = \cos(\phi[n,\alpha])$ and $\upsilon_n = \sin(\phi[n,\alpha])$ are the real and imaginary parts of the signal, respectively. Similarly, the received signal can be represented as $r[n] = \tilde{x}_n + j\tilde{y}_n$ where $\tilde{x}_n$ and $\tilde{y}_n$ are the real and imaginary parts of the received signal, respectively. The log of the conditional probability density function (PDF) of $\mathbf{r} := \{r[0], r[1], \ldots, r[NN_s - 1]\}$ ([11, p. 540]) is given as

$$\log f(\mathbf{r}|\alpha, h) = -N\log(\pi\sigma^2) - \frac{1}{\sigma^2}\sum_{n=0}^{NN_s-1} \times \left[(\tilde{x_n} - \mu_n)^2 + (\tilde{y_n} - \upsilon_n)^2\right].$$

The MCRB [10] can be expressed as

$$\text{MCRB}_h = \frac{1}{E_\alpha[J_h]}, \tag{17}$$

where the expectation is taken over the data probability mass function and J_h is expressed as

$$J_h = -E_{\mathbf{r}|\alpha}\left\{\frac{\partial^2}{\partial h^2}\log f(\mathbf{r}|\alpha, h)\right\} = \frac{2}{\sigma^2}\sum_{n=0}^{NN_s-1}\left[\left(\frac{\partial\mu_n}{\partial h}\right)^2 + \left(\frac{\partial\upsilon_n}{\partial h}\right)^2\right].$$

Computing the partial derivatives, we get $(\partial\mu_n/\partial h)^2 = \beta_n^2\sin^2(\beta_n h)$ and $(\partial\upsilon_n/\partial h)^2 = \beta_n^2\cos^2(\beta_n h)$, where $\beta_n := 2\pi\sum_{i=0}^{m}\alpha_i q_{n-iN_s}$. Thus,

$$J_h = \frac{2}{\sigma^2}\sum_{n=0}^{NN_s-1}\beta_n^2. \tag{18}$$

Separating (18) into cumulative and correlative terms, we can write

$$J_h = \frac{8\pi^2}{\sigma^2}\sum_{n=0}^{NN_s-1}\left[\sum_{i=0}^{m-1}\frac{\alpha_i}{2} + \alpha_m q_{n-mN_s}\right] \times \left[\sum_{j=0}^{m-1}\frac{\alpha_j}{2} + \alpha_m q_{n-mN_s}\right],$$

which can further be represented as

$$J_h = \frac{8\pi^2}{\sigma^2}\sum_{n=o}^{NN_s-1}\Bigg(\underbrace{\frac{1}{4}\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\alpha_i\alpha_j}_{\text{I}} + \underbrace{q_{n-mN_s}^2}_{\text{II}} + \underbrace{(\alpha_m q_{n-mN_s})\sum_{i=0}^{m-1}\alpha_i}_{\text{III}}\Bigg). \tag{19}$$

Now simplifying each term separately, the first term is simplified as

$$\text{I} = \frac{N_s}{4}\sum_{m=1}^{N}(\alpha_0 + \alpha_1 + \ldots + \alpha_{m-1})^2. \tag{20}$$

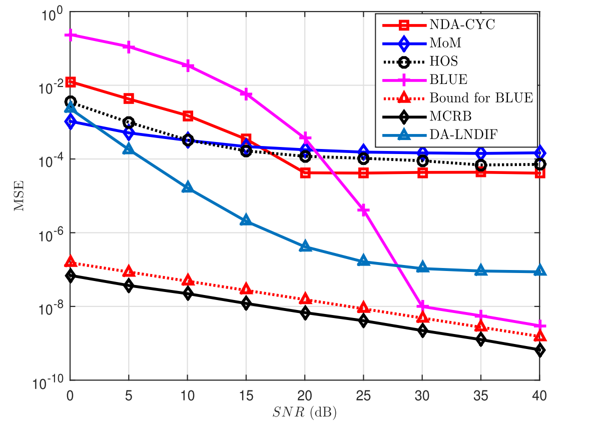

Fig. 1. MSE of different estimators, compared with MCRB vs. SNR with $N = 1000$, $N_s = 4$ and $h = 0.5$.

The second term can be simplified as

$$\text{II} = N\left[q_0^2 + q_1^2 + \ldots. + q_{N_s-1}^2\right] = N\sum_{k=0}^{N_s-1}q_k^2, \tag{21}$$

and similarly, the third term can be simplified as

$$\text{III} = \sum_{k=0}^{N_s-1}q_k\left[\sum_{l=1}^{N}\alpha_l\left(\sum_{p=0}^{l-1}\alpha_p\right)\right]. \tag{22}$$

Substituting (20), (21) and (22) in (19), we get

$$J_h = \frac{8\pi^2}{\sigma^2}\left[\frac{N_s}{4}\left[\sum_{m=1}^{N}(\alpha_0 + \ldots + \alpha_{m-1})^2\right] + N\sum_{k=0}^{N_s-1}q_k^2 + \sum_{k=0}^{N_s-1}q_k\left[\sum_{l=1}^{N_s-1}\alpha_l\left(\sum_{p=0}^{l-1}\alpha_p\right)\right]\right], \tag{23}$$

From (17), we get the expression of MCRB as

$$\text{MCRB}_h = \frac{\sigma^2}{8\pi^2 N}\left[\frac{N_s(N+1)}{8} + \sum_{k=0}^{N_s-1}q_k^2\right]^{-1}. \tag{24}$$

It can be observed that MCRB does not depend upon h and is asymptotically proportional to σ^2, $1/N_s$ and $1/N^2$.

V. Simulations

In this section, we report and analyze the results of numerical simulations and draw a comparison of the performance of various estimators with MCRB in terms of both MSE and BER. The simulations are performed in MATLAB for binary full response CPM signals with rectangular frequency shaping function observed in AWGN. The reported results are obtained by using Monte Carlo simulations averaged over 10^4 trials.

Fig. 1 provides the comparison of MSEs of the proposed estimators, i.e., $\hat{h}_{MoM}$ and $\hat{h}_{BLUE}$ with those of HOS [4], NDA-CYC [5] and DA-LNDIF [3] estimators. The MCRB of (24) and the approximate variance of BLUE given by (16) are also plotted for reference. It can be observed that the non data-aided estimator $\hat{h}_{MoM}$ performs better than the earlier proposed estimators at SNRs below 10 dB and a performance comparable to those of HOS and NDA-CYC estimators for

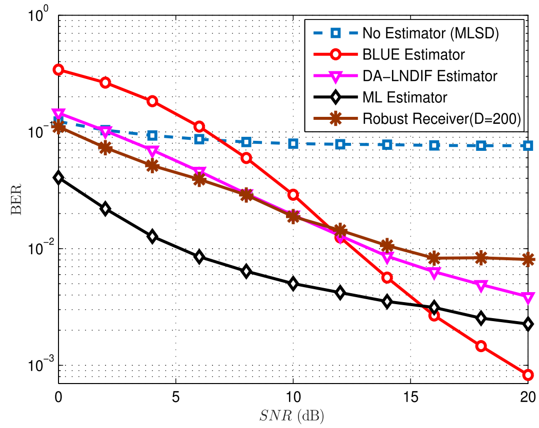

Fig. 2. BER for BLUE, DA-LNDIF [3] and ML [2] estimators and the robust receiver [9] in presence of mismatch of 0.03 using classical 1-REC CPM for $N = 20$, $N_s = 4$ and $h = 1/3$.

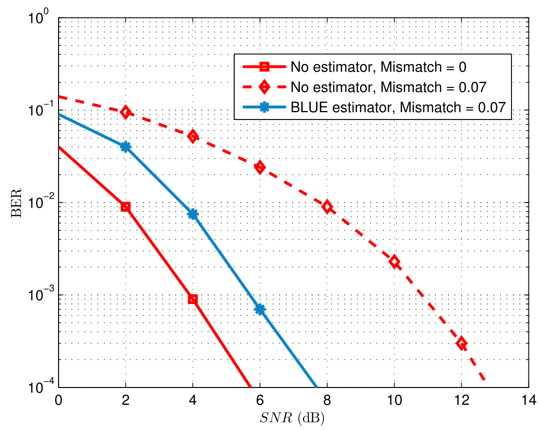

Fig. 3. BER for robust CPM scheme of [8] with modulation index mismatch of 0.07 and performance enhancement with BLUE for $N = 20$, $N_s = 4$ and $h = 0.315$.

moderate and high SNRs. The performance of the NDA-CYC is poor at low SNR due to the phase-unwrapping errors. The performance of data-aided $\hat{h}_{BLUE}$ improves considerably with SNR and approaches the approximate variance bound at SNR of 30 dB. The MSE of the $\hat{h}_{BLUE}$ estimator also exhibits a threshold effect as observed in [11]. Its MSE improves with SNR faster as compared to DA-LNDIF algorithm and approaches close to the MCRB.

Fig. 2 shows the BER performance comparison of the proposed BLUE, DA-LNDIF [3] and ML [2] estimator for the 1-REC CPM with nominal $h = 1/3$. Similarly, the BER performance of the robust receiver [9] is also shown. It can be observed that the BER for the regular maximum-likelihood sequence detection (MLSD) receiver is unacceptably high without the use of estimator, even for a small mismatch of 0.03. The mismatch indicates the difference in the value of modulation index used at the transmitter and the nominal value used at the receiver. The performance of a receiver employing BLUE estimator is better at high SNR as compared to DA-LNDIF and ML (with grid spacing parameter $\mu = 0.01$), while BLUE performance is inferior at low SNR due to phase unwrapping errors. The performance of ML estimator can be further improved by using $\mu = 0.001$ but at the expense of ten times increase in estimator complexity. For the robust receiver, we have considered the discretization parameter $D = 200$ with a uniform prior distribution of h in the range $[0.28, 0.35]$. The robust receiver [9] performance is close to that of DA-LNDIF estimator, which, considering that no estimator is employed, is very reasonable. It should be noted that the BER performance is not linearly proportional to the MSE of estimators because modulation index estimation has a cumulative effect on the receiver performance. Nonetheless, BER follows a direct relationship with MSE. Similar trends in the BER can be observed among non data-aided estimators.

Fig. 3 shows the BER performance for robust modulation schemes proposed in [8]. We observe that even for the robust modulation scheme, when the mismatch value is 0.07, there is a performance loss of around 7 dB at BER of 10^{-4}. Using the BLUE estimator prior to demodulation, we can gain more than 5 dB.

VI. Conclusion

In this letter, we derived the MCRB for a full response CPM signal and proposed two novel estimators for the modulation index. The first estimator is a non data-aided method of moments estimator and the second estimator is the data-aided best linear unbiased estimator. MSE results show that MoM performs better at low SNR than the existing non data-aided estimators and at high SNR, the linear estimator (BLUE) performs closer to the MCRB. Moreover, the BER result shows that the estimators provide performance gain when there is a modulation index mismatch between transmitter and receiver.

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