Novel Construction Methods of Quaternion Orthogonal Designs based on Complex Orthogonal Designs

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***Abstract*—Quaternion orthogonal designs (QODs) are considered the foundation of orthogonal space time polarization block codes (OSTPBCs). OSTPBCs benefit from orthogonal polarizations and orthogonal space and time block coding simultaneously to enhance the capacity of wireless communication systems. To exploit these advantages of OSTPBCs, this paper explores two generalized construction techniques of QODs, where the first one is based on symmetric-paired designs while the second technique maps the complex orthogonal designs (CODs) to QODs directly. With these schemes, QODs for any number of transmit antennas can be constructed. Moreover, a low-complexity maximum-likelihood (ML) decoder for the proposed construction techniques has been presented that provides optimal decoupled decoding with phenomenal complexity reduction. Simulation results show that the diversity order of the first QOD construction is higher than the second design given the number of transmit antennas are same.**

***Index Terms*—Quaternion orthogonal designs (QODs), Maximum-likelihood (ML) decoder, Orthogonal space time polarization block codes (OSTPBCs)**

I. Introduction

Since [1] proposed a way of representing orthogonal polarization states as quaternions, the researchers have started searching ways of constructing orthogonal designs (ODs) over the quaternion domain. In this regard, quaternion orthogonal designs (QODs) are proposed, which are considered the foundation of orthogonal space time polarization block codes (OSTPBCs). OSTPBCs exploit polarization diversity together with space and time diversity to achieve larger transmit diversity gains in communication systems. Complex orthogonal designs (CODs) have been quite successful in elevating the capacity of wireless systems because of their potential to provide full transmit diversity and less decoding delay. To further increase the capacity of wireless systems, one of the pioneering works of QODs, [2], used Alamouti code as a construction unit for designing a QOD for 2×1 dual-polarized antenna-based multiple-input multiple-output (MIMO) system. For this QOD, [2], [3] and [4] derived decoupled decoding rule which was later corrected in [5]. Besides this implementation, [2] has also proposed several construction methods of QODs based on real orthogonal designs, CODs, and pure quaternion-based designs. Furthermore, [6] has proposed a nonlinear method-based QOD construction technique that achieves full rate with reduced complexity of maximum-likelihood (ML) decoding. All these works are based on small scale designs, where the number of transmit and/or receive antennas is small.

For higher order designs, [7] has presented a new class of QODs, called quaternion coordinate interleaved orthogonal designs (QCIODs). These QCIODs are based on coordinate interleaved orthogonal designs (CIODs) and *isomorphism* in which smaller QODs are used to design higher order QODs. However, this construction is limited to even number of transmit antenna arrangements only and 8×8 is the order of highest QCIOD proposed. It is important to note that there are many wireless applications, for instance, MIMO system, opportunistic relaying in wireless sensor networks etc., where the number of transmit antennas may not always be an even number. In this regard, a recent work, [8] explored some constraints in addition to the quaternion orthogonality and symmetricity that must be satisfied by the coding matrix to provide low-complexity decoding. QOD designs for both even and odd constructions are proposed. However, a generalized method to map QODs for higher number of transmit antennas has not been investigated. Therefore, there has been a need to explore generalized QOD constructions given any number of transmit antennas.

Given the amount of work done on QODs, this paper explores two COD-based generalized construction techniques for QODs given any number of transmit antennas. One is based on symmetric-paired designs, whereas second maps existing CODs on QODs directly and also satisfies symmetry property. Besides construction, a generalized decoder for these designs has also been proposed. This ML decoder provides optimal decoupled decoding and therefore, remarkably minimizes system complexity and decoding delay. Simulation results verify the diversity-multiplexing tradeoff trend in QODs. Design that achieves better rate compromises on diversity or vice versa.

The arrangement of paper is as follows. Section II describes our system model and Section III presents two construction techniques of symmetric-paired designs. Next, Section IV derives the ML decoding rule for the proposed construction schemes. Finally, simulation results are provided in Section V.

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II. System Model Based on QODs

We begin with the description of some important properties of quaternions that are significant for the explanation of our work. Basically, a quaternion consists of four real numbers or two complex numbers, for example, $q = q_0 + q_1 i + q_2 j + q_3 k = z_1 + z_2 j$, where $q_o \in \mathbb{R}$; $o = \{0, 1, 2, 3\}$ and $z_p \in \mathbb{C}$; $p = \{1, 2\}$. Moreover, $i^2 = j^2 = k^2 = ijk = -1$ and $ij = k, jk = i, ki = j$ [9]. The quaternion conjugate of q can be written as $q^Q = z_1^* - jz_2^*$ and multiplication of q^Q with q gives $|q|^2$. Similarly, quaternion transpose of a matrix $\mathbf{A} = [a_{r\times n}]$ produces $\mathbf{A}^Q = [a^Q_{n\times r}]$, where $r \times n$ is the dimension of matrix $\mathbf{A}$. Out of all the construction techniques given by [2], we would be focusing on QOD construction based on CODs, therefore, we would define QOD in terms of complex numbers only.

Definition 1.1 (QOD): A QOD $\mathbf{Q}$ on complex variables $\{z_1, z_2, ..., z_u\}$, is a $r \times n$ matrix which can have entries from the set $\{0, z_1, z_1^*, z_2, z_2^*, ..., z_u, z_u^*\}$ including possible multiplications on the left and/or right by quaternion elements $q \in \mathbf{Q}$, and satisfies the following condition,

$$\mathbf{Q}^Q\mathbf{Q} = \sum_{h=1}^{u}(s_h(|z_h|)^2)\mathbf{I}_n = \lambda \mathbf{I}_n. \tag{1}$$

where λ is a positive real number and $\mathbf{I}_{n\times n}$ is the $n \times n$ identity matrix. It is noted that the non-commutative nature of quaternions makes it difficult to generalize CODs to QODs. To construct QODs from CODs in a consistent and transparent manner, we need to define *symmetric-paired* and *complex amicable designs* defined on complex variables.

Definition 1.2. (Symmetric-Paired Designs) Two CODs $\mathbf{A}$ and $\mathbf{B}$ based on complex variables $\{z_1, z_2, ..., z_u\}$ form a symmetric-paired design $(\mathbf{A} + \mathbf{B}j)$ provided $\mathbf{A}^H\mathbf{B}$ or $\mathbf{B}^H\mathbf{A}$ is symmetric.

Definition 1.3. (Complex Amicable Designs) Two CODs $\mathbf{A}$ and $\mathbf{B}$ based on complex variables $\{z_1, z_2, ..., z_u\}$ are called complex amicable designs if $\mathbf{A}^H\mathbf{B} = \mathbf{B}^H\mathbf{A}$ and/or $\mathbf{A}\mathbf{B}^H = \mathbf{B}\mathbf{A}^H$.

We consider a MIMO transmission system with N_t dual-polarized antennas at the transmitter and one dual-polarized antenna at the receiver. The transmitting antennas transmit data using OSTPBCs, which are received by the receiving antenna in the form of quaternion vectors. The channel matrix is represented as $\mathbf{H}^{(m)} = \begin{bmatrix} h_{11}^{(m)} & h_{12}^{(m)} \\ h_{21}^{(m)} & h_{22}^{(m)} \end{bmatrix}$, where $m = \{1, 2, ..., N_t\}$. We assume a Rayleigh fading channel, therefore, each term of channel gain matrix represents a complex Gaussian random variable (RV) with zero mean and unit variance. For this $N_t \times 1$ antenna arrangement, the received signal $\mathbf{R}$ can be written as

$$\mathbf{R} = \mathcal{C}(\mathbf{Q}) \begin{bmatrix} \mathbf{H}^{(1)} \\ \vdots \\ \mathbf{H}^{(N_t)} \end{bmatrix} + \begin{bmatrix} n_{11} & n_{12} \\ \vdots & \vdots \\ n_{N_t 1} & n_{N_t 2} \end{bmatrix}, \tag{2}$$

where the entries $n_{k_1 k_2}\ \forall\ k_1 = \{1, 2, ..., N_t\}, k_2 = \{1, 2\}$ denote the elements of white noise as two dimensional independent and identically distributed (i.i.d.) complex Gaussian RVs with zero mean and identical variance per dimension, while $\mathcal{C}(\mathbf{Q})$ is a complex code matrix containing transmitted symbols. $\mathbf{R}$ represents a quaternion vector, i.e., $\mathbf{R} = \mathcal{C}^{-1}\{\mathcal{C}(\mathbf{Q})\mathbf{H} + \mathbf{N}\}$, where $\mathbf{N}$ represents the noise matrix. The operator $\mathcal{C}^{-1}\{.\}$ and its counterpart $\mathcal{C}(.)$ operates on complex numbers z_1, z_2 as $\mathcal{C}^{-1}\{z_1, z_2\} = z_1 + z_2 j$ and $\mathcal{C}(z_1 + z_2 j) = [z_1, z_2]$. In this paper, we will be using $\mathcal{C}^{-1}\{\mathbf{Q}\}$ and $\mathbf{Q}$ interchangeably.

III. Construction Techniques

In this section, we present two generalized construction techniques of symmetric-paired designs.

A. Symmetric-Paired Design 1:

According to the Definition 1.2, a symmetric-paired design $\mathbf{A} + \mathbf{B}j$ yields a valid QOD only if CODs $\mathbf{A}$ and $\mathbf{B}$ satisfy the symmetry property. Symmetric-paired design 1 construction technique covers those symmetric-paired designs in which COD $\mathbf{B}$ is obtained from $\mathbf{A}$ through permutation of columns, where permutation operation on two columns i and j results in swapping the positions of these two columns with each other. It is important to note that as the dimension of COD matrix gets larger, not every permutation of columns of a COD $\mathbf{B}$ yields a valid QOD.

Essentially, $\mathbf{A}^H\mathbf{B} = (\mathbf{A}^H\mathbf{B})^T$ translates to the condition that in matrix $\mathbf{A}^H\mathbf{B}$, each $\alpha_{i,j} = \alpha_{j,i}$, where $\alpha_{i,j}$ denotes the element present at the ith row and jth column of matrix $\mathbf{A}^H\mathbf{B}$. By the definition of COD, we know that $\mathbf{A}^H\mathbf{A} = \lambda\mathbf{I}$. Therefore, if COD $\mathbf{B}$ is same as COD $\mathbf{A}$, then $\mathbf{A}^H\mathbf{B} = \lambda\mathbf{I}$. However, if permutation operation is performed on columns j and k of COD $\mathbf{B}$, then both entries, i.e., $\alpha_{j,k}$ and $\alpha_{k,j}$ of $\mathbf{A}^H\mathbf{B}$ will become λ. Afterwards, if column k gets permuted with any other column, then $\alpha_{j,k}$ and $\alpha_{k,j}$ will no longer be equal and consequently, will not satisfy symmetry property. That is why, only one permutation per column is allowed. As the diversity order of all valid designs constructed from the permutation of columns of a COD remains the same, therefore, any one of them can be picked to give a closed-form of symmetric-paired QODs. In [10], a general COD is designed for $l + 1$ symbols embedded in a square matrix of order 2^l such that

$$\mathbf{A} = \begin{bmatrix} \mathbf{G}_{2^{l-1}}(z_1, z_2, .., z_l) & z_{l+1}\mathbf{I}_{2^{l-1}} \\ -z_{l+1}^*\mathbf{I}_{2^{l-1}} & \mathbf{G}^H_{2^{l-1}}(z_1, z_2, .., z_l) \end{bmatrix}, \tag{3}$$

where $l = \{1, 2, 3...\}$ and $\mathbf{G}_{2^{l-1}}(z_1, z_2, .., z_l)$ represents a COD of order $2^{l-1} \times 2^{l-1}$ defined on symbols $z_1, z_2, .., z_l$. For example, for $l = 1$, $\mathbf{G}_1(z_1) = [z_1]$. Moreover, $\mathbf{G}_2(z_1, z_2)$, $\mathbf{G}_3(z_1, z_2, z_3)$ and $\mathbf{G}_4(z_1, z_2, z_3, z_4)$ are given in [10]. We now use the above scheme to generate square QODs. In this construction, columns $1, 2, .., N_t/2$ of matrix $\mathbf{A}$ are swapped with $(N_t/2) + 1, (N_t/2) + 2, .., N_t$ columns, respectively, to give matrix $\mathbf{B}$, where N_t represents the number of antennas of COD on which permutation is performed. We now arrive at the following result where we have omitted the redundant argument in $\mathbf{G}$.

Theorem 3.1. *For a given COD $\mathbf{A}$ in (3), a complex amicable and symmertic-paired design can be constructed such that the*

following realization
$\mathbf{Q}_{2^l}(z_1, z_2, .., z_{l+1}) = \mathbf{A} + \mathbf{B}j =$

$$\begin{bmatrix} \mathbf{G}_{2^{l-1}} + z_{l+1}\mathbf{I}_{2^{l-1}}j & z_{l+1}\mathbf{I}_{2^{l-1}} + \mathbf{G}_{2^{l-1}}j \\ -z_{l+1}^*\mathbf{I}_{2^{l-1}} + \mathbf{G}_{2^{l-1}}^H j & \mathbf{G}_{2^{l-1}}^H - z_{l+1}^*\mathbf{I}_{2^{l-1}}j \end{bmatrix}, \quad (4)$$

provides a QOD of dimension $2^l \times 2^l$ *with rate* $(l+1)/2^l$.

Proof. We begin by proving that both $\mathbf{A}$ and $\mathbf{B}$ in (4) satisfy the symmetry and amicability property. We first observe that

$$\mathbf{A}_{2^l}^H \mathbf{B}_{2^l} = \begin{bmatrix} \mathbf{O}_{2^{l-1}} & 2\lambda_1\mathbf{I}_{2^{l-1}} \\ 2\lambda_1\mathbf{I}_{2^{l-1}} & \mathbf{O}_{2^{l-1}} \end{bmatrix}, \quad (5)$$

where $\lambda_1\mathbf{I}_{2^{l-1}} = \mathbf{G}_{2^{l-1}}^H\mathbf{G}_{2^{l-1}} + z_{l+1}z_{l+1}^*\mathbf{I}_{2^{l-1}}$ and $\mathbf{O}_{2^{l-1}}$ is a null matrix of order $2^{l-1} \times 2^{l-1}$. It can be seen from Eq. (5) that $\mathbf{A}^H\mathbf{B}$ is symmetric as its transpose will not change this expression. Moreover, it also satisfies complex amicable design condition as $\mathbf{B}^H\mathbf{A}$ is equal to $\mathbf{A}^H\mathbf{B}$ given in (5). Now, to prove quaternion orthogonality, quaternion conjugate of (4) can be written as
$\mathbf{Q}_{2^l}^Q(z_1, z_2, .., z_{l+1}) =$

$$\begin{bmatrix} \mathbf{G}_{2^{l-1}}^H - jz_{l+1}^*\mathbf{I}_{2^{l-1}} & -z_{l+1}\mathbf{I}_{2^{l-1}} - j\mathbf{G}_{2^{l-1}} \\ z_{l+1}^*\mathbf{I}_{2^{l-1}} - j\mathbf{G}_{2^{l-1}}^H & \mathbf{G}_{2^{l-1}} + jz_{l+1}\mathbf{I}_{2^{l-1}} \end{bmatrix}. \quad (6)$$

Following the definition of QOD, multiplication of (6) with (4) provides

$$\mathbf{Q}_{2^l}^Q\mathbf{Q}_{2^l} = \begin{bmatrix} 2\lambda_1\mathbf{I}_{2^{l-1}} & \mathbf{O}_{2^{l-1}} \\ \mathbf{O}_{2^{l-1}} & 2\lambda_1\mathbf{I}_{2^{l-1}} \end{bmatrix} = \lambda\mathbf{I}_{2^l}, \quad (7)$$

where $\lambda = 2\lambda_1$. Hence, Theorem 3.1 is proved. □

In order to illustrate the construction, we present the following example.

Example 1. Suppose we have a rate one Alamouti code $\mathbf{G}_2 = \begin{bmatrix} z_1 & z_2 \\ -z_2^* & z_1 \end{bmatrix}$, then using (3) we first obtain a COD of order 4. The matrix $\mathbf{B}$ is generated using (4) and finally we get the following QOD for 4 dual-polarized antennas

$$\mathbf{Q} = \begin{bmatrix} z_1 + z_3j & z_2 & z_3 + z_1j & z_2j \\ -z_2^* & z_1^* + z_3j & -z_2^*j & z_3 + z_1^*j \\ -z_3^* + z_1^*j & -z_2j & z_1^* - z_3^*j & -z_2 \\ z_2^*j & -z_3^* + z_1j & z_2^* & z_1 - z_3^*j \end{bmatrix}.$$

With this QOD, 3 complex symbols z_1, z_2 and z_3 can be transmitted in four time slots and therefore, it provides 3/4 code rate. It is important to note that (4) formulates QODs for only those systems where number of dual-polarized transmit antennas can be written in a power of 2. For other dual-polarized transmit antenna arrangements, truncation of unnecessary columns provides the desired QOD.

B. Symmetric-Paired Design 2:

In the first construction scheme, the matrix $\mathbf{B}$ is obtained from $\mathbf{A}$ by permuting its columns. However, a general mechanism is required to generate quaternion designs from two arbitrary CODs. We propose a rather simple construction to generate such designs and prove that they perform relatively better than the designs obtained from the first technique. To the best knowledge of the authors, this construction has not been developed before. It maps existing CODs to QODs directly.

Lemma 3.1. *For a given square COD* $\mathbf{G}_{2^{l-1}}(z_1, z_2, .., z_{l+1})$, *the matrix*
$\mathcal{Q}_{2^1 \times 2^{l-1}}(z_1, z_2, .., z_{l+1}) =$

$$\begin{bmatrix} \mathbf{G}_{2^{l-1}}(z_1, z_2, .., z_l) + z_{l+1}\mathbf{I}_{2^{l-1}}j \\ -z_{l+1}^*\mathbf{I}_{2^{l-1}} + \mathbf{G}_{2^{l-1}}^H(z_1, z_2, .., z_l)j \end{bmatrix} \quad (8)$$

provides a quaternion design of order $2^l \times 2^{l-1}$, *with rate* $(l+1)/2^l$.

Proof. We first note that for this design, 0 is the only element present in $\mathbf{A}^H\mathbf{B}$ as each column of matrix $\mathbf{A}$ is orthogonal to matrix $\mathbf{B}$ and vice versa. Therefore, it is a symmetric-paired and a complex amicable design. Following the steps of proof of Theorem 3.1, it is straightforward to show that

$$\mathbf{Q}_{2^{l-1}\times 2^l}^Q\mathbf{Q}_{2^l\times 2^{l-1}} = 2\lambda_1\mathbf{I}_{2^{l-1}} = \lambda\mathbf{I}_{2^{l-1}}. \quad (9)$$

which complete the proof of Lemma 3.1. □

Hence, this construction also provides valid QODs as is also evident from the following example.

Example 2. Using (8), we get the following QOD for 2 dual-polarized antennas

$$\mathbf{Q} = \begin{bmatrix} z_1 + z_3j & z_2 \\ -z_2^* & z_1^* + z_3j \\ -z_3^* + z_1^*j & -z_2j \\ z_2^*j & -z_3^* + z_1j \end{bmatrix}.$$

With this QOD, 3 complex symbols z_1, z_2 and z_3 can be transmitted in four time slots and therefore, it provides 3/4 data rate. It is important to note that symmetric-paired design 1 provides minimum decoding delay as compare to symmetric-paired design 2. That is because design 2 transmits data in comparatively more number of time slots, for example, for 2 dual-polarized antennas, design 2 provides a QOD of order 4×2 and design 1 yields a QOD of order 2×2. Expression (8) generates QODs for system arrangements in which number of transmit antennas can be represented in a power of 2 and for other arrangements, truncation of unnecessary columns results in the desired QOD.

IV. Low-Complexity Decoder

For QODs, ML decoding rule corresponds to the minimization of either the norm $\|\mathbf{R} - \mathcal{C}^{-1}\{\mathcal{C}(\mathbf{Q})\mathbf{H}\}\|$ or its square for finding the transmitted symbols. The squared norm can be given as,

$$\begin{aligned} &\min\left(\|\mathbf{R} - \mathcal{C}^{-1}\{C(\mathbf{Q})\mathbf{H}\}\|^2\right) = \\ &\min\, tr\left((\mathbf{R} - \mathcal{C}^{-1}\{\mathcal{C}(\mathbf{Q})\mathbf{H}\})^Q(\mathbf{R} - \mathcal{C}^{-1}\{\mathcal{C}(\mathbf{Q})\mathbf{H}\})\right) = \\ &\min\, \underbrace{tr\left(\mathbf{R}^Q\mathbf{R}\right)}_{\Gamma^{(1)}} \underbrace{-2\Re\Big(tr\left(\mathbf{R}^Q\left(\mathcal{C}^{-1}(\mathcal{C}(\mathbf{Q})\mathbf{H})\right)\right)\Big)}_{\Gamma^{(2)}} \\ &\underbrace{+tr\left(\left(\mathcal{C}^{-1}\left(\mathcal{C}(\mathbf{Q})\mathbf{H}\right)\right)^Q\left(\mathcal{C}^{-1}\left(\mathcal{C}(\mathbf{Q})\mathbf{H}\right)\right)\right)}_{\Gamma^{(3)}}, \end{aligned} \quad (10)$$

where $\Re(.)$ and $tr(.)$ signify the real part of a complex number and trace operator, respectively. In (10), the superscript k of $\Gamma^{(k)}$ represents the index of corresponding term in the decoding statistics, respectively. Our aim is to simplify (10) to design low complexity decoder. This is proven in the following lemma.

Lemma 5.1 *The ML decoding rule derived for any QOD, comprises of three real valued elements* $\Gamma^{(1)}$, $\Gamma^{(2)}$ *and* $\Gamma^{(3)}$ *corresponding to the norms of received signal vector* $\mathbf{R}$, *mixed transmitted symbol and channel coefficients vector* $\mathcal{C}^{-1}\{\mathcal{C}(\mathbf{Q})\mathbf{H}\}$, *respectively*

$$\Gamma^{(1)} = \sum_{i=1}^{t} |r_i|^2, \; \Gamma^{(3)} = \sum_{i=1}^{t}\sum_{k=1}^{2N_t} |\alpha_{ik}(\beta_{k1} + \beta_{k2}j)|^2, \tag{11}$$

$$\Gamma^{(2)} = -2\Re\Big(\sum_{i=1}^{t}\sum_{k=1}^{2N_t}(r_i^Q \alpha_{ik}(\beta_{k1} + \beta_{k2}j)\Big), \tag{12}$$

where t *refers to time slots. Moreover,* r_x, α_{xy} *and* β_{xy} *represent elements present in received quaternion vector, complex code matrix and channel matrix at row* x *and column* y, *respectively.*

Proof. The term $\mathcal{C}(\mathbf{Q})\mathbf{H}$ can be obtained by simple matrix multiplication and result of it will always be a matrix of $N_t \times 2$ dimension for our system model. Application of $\mathcal{C}^{-1}\{.\}$ operator on this expression converts it to a vector of order $N_t \times 1$. Likewise, it can be seen from Eq. (2) that $\mathbf{R}$ is a quaternion vector of length N_t and its quaternion transpose changes its order to $1 \times N_t$. Thus, multiplication of $\mathbf{R}^Q$ with $\mathcal{C}^{-1}\{\mathcal{C}(\mathbf{Q})\mathbf{H}\}$ provides only one quaternion element. With the help of basic quaternion arithmetic, it can be seen that the expression (12) represents the simplified form of that quaternion element. According to quaternionic algebra, multiplication of $\mathbf{x}^Q$ with $\mathbf{x}$ is equal to the sum of norm squared of each quaternion element present in the quaternion vector $\mathbf{x}$. With this rule, $\Gamma^{(1)}$ and $\Gamma^{(3)}$ come out to be the sum of norm squared of each quaternion element present in $\mathbf{R}$ and $\mathcal{C}^{-1}\{\mathcal{C}(\mathbf{Q})\mathbf{H}\}$, respectively, as given in (11). □

Now, we derive low-complexity decoder for the symmetric-paired design constructions presented in (4) and (8).

Lemma 5.2: *The ML decoding rule for the symmetric-paired design constructions given in (4) and (8), simplifies to the following form,*

$$\min\left(\|\boldsymbol{R} - \mathcal{C}^{-1}\{\mathcal{C}(\boldsymbol{Q})\boldsymbol{H}\}\|^2\right) = \min\left(\lambda \, tr(\tilde{\mathbf{H}}) + \Gamma^{(2)}\right), \tag{13}$$

where $\tilde{\mathbf{H}}$ *contains mixed combinations of complex channel coefficients only.*

Proof. $\Gamma^{(1)}$ is independent of transmitted symbols and does not contribute in the norm calculation, hence, it can be neglected. Similarly, $\Gamma^{(2)}$ is a linear combination of transmitted symbols and can be decoupled easily. However, there is a need to expand $\Gamma^{(3)}$ to analyze its impact on decoding statistics. Let $\mathbf{h}_1$ and $\mathbf{h}_2$ denote the column vectors of the channel matrix, then the term $\Gamma^{(3)}$ can be simplified

$$\begin{aligned}\Gamma^{(3)} &= tr\Big((\mathcal{C}(\mathbf{Q})\mathbf{h}_1 + \mathcal{C}(\mathbf{Q})\mathbf{h}_2 j)^Q(\mathcal{C}(\mathbf{Q})\mathbf{h}_1 + \mathcal{C}(\mathbf{Q})\mathbf{h}_2 j)\Big)\\ &\overset{(a)}{=} tr\Big(\mathbf{h}_1^H(\mathcal{C}(\mathbf{Q}))^H\mathcal{C}(\mathbf{Q})\mathbf{h}_1\Big) + tr\Big(\mathbf{h}_2^H(\mathcal{C}(\mathbf{Q}))^H\mathcal{C}(\mathbf{Q})\mathbf{h}_2\Big)\\ &\overset{(b)}{=} \lambda tr\Big(\mathbf{h}_1^H\mathbf{E}\mathbf{h}_1 + \mathbf{h}_2^H\mathbf{E}\mathbf{h}_2\Big) = \lambda \, tr(\tilde{\mathbf{H}}).\end{aligned} \tag{14}$$

It is easy to prove that amicability property implies $(\mathcal{C}(\mathbf{Q}))^H\mathcal{C}(\mathbf{Q}) = \lambda\mathbf{E}$ in symmetric-paired QODs, where $\mathbf{E}$ is a symmetric matrix and contains zeros and ones only. It may not necessarily be an identity matrix. (a) follows from commutativity of complex numbers and (b) is the result of application of amicability property of the proposed symmetric-paired construction techniques. Hence, this simplification leads to the decoding rule (13). □

The following corollary further reduces the number of calculations of the decoding rule (13).

Corollary 5.1. *Use of quadrature phase shift keying (QPSK) modulation for the transmission of the symmetric-paired design constructions given in (4) and (8) further simplifies the decoding rule (13) to*

$$\min\left(\|\mathbf{R} - \mathcal{C}^{-1}\{\mathcal{C}(\mathbf{Q})\mathbf{H}\}\|^2\right) = \min(\Gamma^{(2)}), \tag{15}$$

where (15) comes from the fact that the symbols are of equal energy and that the minimization is not affected by channel coefficients.

Now, we will use decoding rule (15) to decode the transmitted symbols of QODs given in Examples 3.1 and 3.2.

Corollary 5.2. *The decoupled decoding statistics for each transmitted symbol* z_1, z_2 *and* z_3 *for QOD in Example 3.1 is*

$$\begin{aligned}&\min_{z_1} -2\Re\{r_1^Q z_1 g_{12} + r_2^Q z_1^* g_{34} + r_3^Q z_1^* g_{56} + r_4^Q z_1 g_{78}\},\\ &\min_{z_2} -2\Re\{r_1^Q z_2 g_{34} - r_2^Q z_2^* g_{12} - r_3^Q z_2 g_{78} + r_4^Q z_2^* g_{56}\},\\ &\min_{z_3} -2\Re\{r_1^Q z_3 g_{56} + r_2^Q z_3 g_{78} - r_3^Q z_3^* g_{12} - r_4^Q z_3^* g_{34}\},\end{aligned} \tag{16}$$

respectively, in which $g_{mn} = g_m + g_n j$ *for all* m, n. *Furthermore,* $g_1 = h_{11}^{(1)} + h_{21}^{(3)}$, $g_2 = h_{12}^{(1)} + h_{22}^{(3)}$, $g_3 = h_{11}^{(2)} + h_{21}^{(4)}$, $g_4 = h_{12}^{(2)} + h_{22}^{(4)}$, $g_5 = h_{21}^{(1)} + h_{11}^{(3)}$, $g_6 = h_{22}^{(1)} + h_{12}^{(3)}$, $g_7 = h_{21}^{(2)} + h_{11}^{(4)}$ *and* $g_8 = h_{22}^{(2)} + h_{12}^{(4)}$.

Corollary 5.3. *The decoupled decoding statistics for each transmitted symbol* z_1, z_2 *and* z_3 *for QOD in Example 3.2 is*

$$\begin{aligned}&\min_{z_1} -2\Re\{r_1^Q z_1 e_1 + r_2^Q z_1^* e_2 + r_3^Q z_1^* e_3 + r_4^Q z_1 e_4\},\\ &\min_{z_2} -2\Re\{r_1^Q z_2 e_2 - r_2^Q z_2^* e_1 - r_3^Q z_2 e_4 + r_4^Q z_2^* e_3\},\\ &\min_{z_3} -2\Re\{r_1^Q z_3 e_3 + r_2^Q z_3 e_4 - r_3^Q z_3^* e_1 - r_4^Q z_3^* e_2\}\end{aligned} \tag{17}$$

respectively, where $e_1 = h_{11}^{(1)} + h_{12}^{(1)}j$, $e_2 = h_{11}^{(2)} + h_{12}^{(2)}j$, $e_3 = h_{21}^{(1)} + h_{22}^{(1)}j$ *and* $e_4 = h_{21}^{(2)} + h_{22}^{(2)}j$.

V. Simulation Results

We have compared the performance of both the designs by constructing QODs for 2, 3 and 4 dual polarized transmit antennas that include both even and odd antenna arrangements. Their decoding statistics are derived using Lemma 3.2 and are completely decoupled. The rate of design 1 for these antenna arrangements is 1, 3/4 and 3/4, respectively. Likewise, code rate of design 2 is 3/4, 1/2 and 1/2, respectively. For simulations, we have considered QPSK constellation. The channel coefficients are assumed perfectly known at the receiver, whereas, white noise is added in each polarization uniformly.

Fig. 1 presents the simulation results of QOD for 2, 3 and 4 dual-polarized transmit antennas for both the designs. As symmetric-paired design 1 provides a QOD similar to the one presented in [2], therefore, its bit error rate (BER) curve is identical to [2]. This similarity hints the accuracy of our work. Overall, system's diversity gain increases with each added antenna dimension. For example, for symmetric-paired design 1, 10^{-5} BER is achieved by 2Tx/1Rx, 3Tx/1Rx and 4Tx/1Rx antenna arrangements at signal-to-noise ratio (SNR) values 16, 11.8 and 9.4 dB, respectively. Similarly, for symmetric-paired design 2, the same BER is achieved by 2T/1Rx, 3Tx/1Rx and 4Tx/1Rx antenna arrangements at 12.2, 9.8 and 8 dB SNR values, respectively. Furthermore, impact of the transmission rate of codes can also be observed from Figure 1. For both designs, the SNR gain from 2Tx/1Rx to 3Tx/1Rx is larger than from 3Tx/1Rx to 4Tx/1Rx. This is because for the former case, rate has dropped from 1 to 3/4 for design 1 and from 3/4 to 1/2 for design 2. However, in the latter scenario, the rate remains the same for both designs.

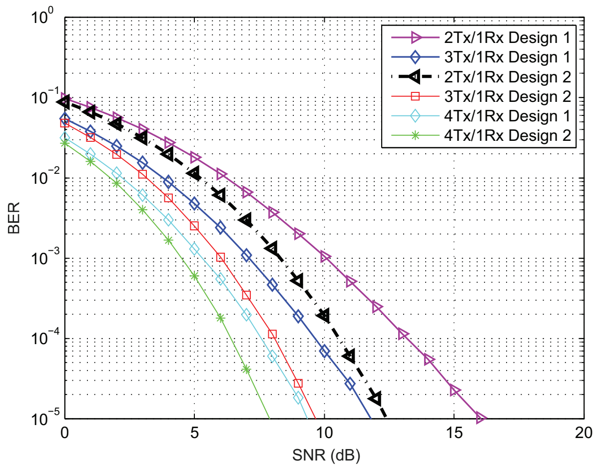

Figure 1: BER vs. SNR for OSTPBCs with 2, 3 and 4 Tx antennas.

In a nutshell, the trend of BER curves indicates that design 2 performs better than design 1 in terms of diversity gain because each BER curve of design 2 provides lesser BER as compared to its respective design 1. However, as antenna dimensions increase, this improvement starts decreasing. That is because for higher order designs, symmetric-paired design 2 provides a larger QOD comprising of more number of zeros than its respective symmetric-paired design 1. For example, for 5Tx/1Rx case, symmetric-paired design 2 produces a QOD of order 16×5 which transmits 5 symbols in 16 time slots whereas symmetric-paired design 2 generates a QOD of order 8×5 which transmits 4 symbols in 8 timeslots. This shows that for higher order designs, former design performance is better in terms of rate and decoding delay with a slight compromise on diversity. On the other hand, symmetric-paired design 2 is suitable for small scale designs only because for higher order designs, its rate falls below half and therefore, its fails to provide high code rates.

VI. Conclusion and Future Works

This paper presented two generalized construction techniques of so called OSTPBCs for data transmission in MIMO systems comprising dual-polarized antennas. These generalized schemes can construct QOD for any number of transmit antennas. A ML decoder has also been derived that provides decoupled decoding at the receiver side and therefore, reduces decoding complexity significantly. It has been verified through simulation results that the symmetric-paired design 2 outperforms the first one in terms of transmit diversity. On the other hand, symmetric-paired design 1 performs better than design 2 in terms of data rate. Hence, there is a tradeoff between data rate and transmit diversity.

In future, we intend to explore pure quaternion-based QOD design constructions. Moreover, we would like to explore QODs providing higher transmit diversity to multiply the benefits of QODs. In addition to this, we also aim to investigate the design of a generalized low-complexity decoupled decoder that can provide decoupled decoding for any QOD without any other constraints.

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