Efficient Quaternion-Based Fast-Decodable Space Time Codes

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***Abstract*—Design of a low-complexity optimal decoder for quaternion orthogonal designs (QODs) has been an open research problem. In this paper, we identify the main issue with the conventional maximum-likelihood (ML) decoder based on quaternion-norm that obstructs the existence of decoupled decoding at the receiver. An optimal decoder which has both characteristics, i.e., linear and decoupled, is found using transmit precoding to realize an effective channel at the receiver that simplifies the criterion and ultimately leads to a low-complexity decoder for all QODs. The proposed solution is then successfully applied to a wide range of quaternion designs where the conventional ML decoder fails to provide decoupled decoding. Simulation results of these designs validate the optimality of the proposed generalized decoder. Overall, the bit error rate (BER) curves exhibit the trend of diversity-multiplexing tradeoff, i.e., the coded matrices of low code rate outperform the ones of higher code rate in terms of diversity.**

***Index Terms*—Quaternion orthogonal designs, maximum-likelihood decoder, quasi space time block codes.**

I. Introduction

One of the main motivations behind the use of complex orthogonal space time block codes (COSTBCs) has been their orthogonality property that simplifies the maximum-likelihood (ML) decoding rule by providing decoupled linear equations at the receiver end. Coding matrices often achieve this attractive feature of low-complexity decoder by compromising on the code rates, i.e., the ratio of the number of independent complex transmitted symbols and the number of total time slots taken to transmit a coding matrix. It is important to note that the full-rate COSTBCs only exist for 2 transmit antennas and maximum code rate of COSTBCs approaches half with the increase in the number of transmit antennas [1]. Various designs have been explored such as quasi-orthogonal space time block codes (QOSTBCs) to provide comparatively higher code rates but these designs result in either pair-wise or completely coupled ML decoding [2]. To overcome this performance barrier, [4] proposed the use of quaternion orthogonal designs (QODs), which laid the foundation of orthogonal space time polarization block codes (OSTPBCs). A promising feature of these designs has been their decoupled decoding for any QOD [4].

Since Isaeva and Sarytchev presented a way to represent a signal from a dual-polarized transmit antenna as a quaternion in [3], [4] laid the foundation of OSTPBCs that utilize polarization together with space and time diversity with the help of quaternion algebra. [4] presented various QOD construction techniques and showed with an example of 2×2 order QOD that quaternion decoding statistics can provide decoupled decoding for any QOD. Their subsequent works [5]-[6] used the same QOD and emphasized the similar postulate that quaternion ML norm can provide optimal decoupled decoding for any QOD construction. Later on, the authors corrected their decoding rule in [7] and highlighted that the proposed decoding rule does not yield optimal decoding for all QODs and therefore, the design of a semi-optimal or optimal low-complexity decoder for QODs remains an open research problem [8]. To address this issue, recent works, [9] and [10], presented two generalized QOD construction techniques, which, however, focused on QODs comprised of either orthogonal or same complex vectors. In an another work, it is emphasized that quaternion domain naturally describe point-to-point transmission of information among dual-polarized antennas exploiting polarization diversity from non-zero cross polarization terms [11]. However, the design of a generalized ML decoder remains a major research problem in the considered area.

Given the progress on QODs, this paper is the first work that seeks to solve the open research problem of decoupled decoding of QODs [8] by proposing a generalized optimal decoder. We begin by presenting various examples of QODs that do not have optimal linear decoding solutions with the previous ML decoding schemes [4], [9]. We explore the primary reason of failure of applicability of quaternion orthogonality constraint on the quaternion ML norm for providing a decoupled decoding for any QOD. As a solution, we precode each transmitted symbol of QODs before transmission. This helps in modifying the impact of a dual-polarized transmission channel at the receiver side in such a way that the ML quaternion norm criterion simplifies to a decoupled linear equation-based decoding solution, which reduces the decoder complexity significantly.

The arrangement of paper is as follows. Section II describes different categories of QODs and Section III highlights the root cause of failure of quaternion norm criterion-based ML decoder. Next, Sections IV and V present the proposed system model and decoder, respectively. Finally, simulation results are provided in Section VI.

II. Quaternion Orthogonal Designs

It was proposed in [3] that a signal transmitted in two orthogonal polarizations can be expressed as a quaternion [12],

[3]. Consequently, signal's complex constellation also become orthogonal in polarization plane. A primary purpose of this quaternionic representation is the simplification of mathematical algorithms devised for the analysis of polarizations. Later on, [4] showed that this representation of a polarized signal as a combination of quaternion amplitude and quaternion phase rotation helps in exploring the polarization diversity together with other classes of diversities such as space and time diversity in orthogonal codes. Essentially, a quaternion comprises of four real numbers, i.e., $q = r_0 + r_1 i + r_2 j + r_3 k$ such that $r_i \in \mathbb{R}$, $i^2 = j^2 = k^2 = ijk = -1$ and $ij = k, jk = i, ki = j$, respectively. Therefore, the space of quaternions is isomorphic to $\mathbb{R}^4$. A quaternion can be formed either by four real numbers, e.g., $q = r_0 + r_1 i + r_2 j + r_3 k$ or by two complex numbers, e.g., $q = z_1 + z_2 j$, where z_1 and z_2 are two complex numbers. Although [4] has defined QODs on real, complex and quaternion elements, we state the definition of QOD based on quaternion numbers as it is more generalized.

Definition 1.1 (QOD): A QOD $\mathbf{Q}$ on quaternion variables $\{q_1, q_2, ..., q_u\}$ of type $\{s_1, s_2, ..., s_u\}$ is a $r \times n$ matrix which can have entries from the set $\{0, q_1, q_1^Q, q_2, q_2^Q, ..., q_u, q_u^Q\}$ including possible multiplications on the left and/or right by quaternion elements $q \in \mathbf{Q}$, and satisfying the condition that,

$$\mathbf{Q}^Q\mathbf{Q} = \sum_{h=1}^{u}(s_h(|q_h|)^2)\mathbf{I}_n = \lambda\mathbf{I}_n, \tag{1}$$

where λ is a positive real number and $\mathbf{I}_{n\times n}$ is the $n \times n$ identity matrix. Moreover, the operator $(.)^Q$ denotes the quaternion transpose and we refer [12] for the explanation of operations of quaternion algebra. Throughout this paper, the operator $\mathcal{C}(.)$ and its counterpart $\mathcal{C}^{-1}(.)$ when applied on a quaternion $(z_1 + z_2 j)$ or complex numbers z_1, z_2 produces $\mathcal{C}(z_1 + z_2 j) = [z_1, z_2]$ and $\mathcal{C}^{-1}\{z_1, z_2\} = z_1 + z_2 j$, respectively. Moreover, operator $(.)^T$, $(.)^H$ and $tr(.)$ refer to the transpose, Hermitian transpose and trace operator, respectively.

It is important to note that the essential orthogonality condition (1) alone has been insufficient in providing an optimal decoder for QODs. Moreover, non-commutative nature of quaternions have made it difficult to generalize COD construction techniques to QODs. Therefore, various sub-categories of QODs have been explored to ease QOD constructions and derive decoding statistics accordingly.

A. QODs based on complex orthogonal designs

In general, there are two ways of generating QODs from real orthogonal designs or CODs that have been explored in literature [4] and we categorize them with respect to a symmetricity property which is briefly described below.

1) Symmetric-Paired Designs. Two CODs $\mathbf{A}$ and $\mathbf{B}$ based on complex variables $\{z_1, z_2, ..., z_u\}$ form a symmetric-pair design $\mathbf{Q} = \mathbf{A} + \mathbf{B}j$, provided $\mathbf{A}^H\mathbf{B}$ or $\mathbf{B}^H\mathbf{A}$ is symmetric. A large class of QODs based on above symmetric-paired designs have been investigated over the years [4]-[6] and [9] in which low-complexity decoders are available but for some specific constructions especially when matrix $\mathbf{B}$ is a column permuted version of $\mathbf{A}$. As it is shown in [8] that there are still many symmetric-paired designs which do not yield a linear decoupled decoding solution at the receiver. Therefore, symmetric-paired designs with respect to ML decoding statistics can be further classified into two categories, i.e., whether QOD is attained by the permutation of columns of the matrix $\mathbf{A}$ or not.

1a. Permuted Designs: These designs are constructed on the technique in which the matrix $\mathbf{B}$ is a column permuted version of the matrix $\mathbf{A}$, which are extensively used following the work of [4]. The same technique was followed up in [9], to construct QODs for higher order transmit antenna arrangements. The authors also showed that these designs satisfy the constraint $\mathbf{C}^H\mathbf{C} = \lambda\mathbf{E}$, where $\mathbf{C} = \mathcal{C}(\mathbf{Q})$ and $\mathbf{E}$ is a symmetric matrix and comprises of ones and zeros only. A simple example illustrating the effectiveness of this line of approach is

$$\mathbf{Q}_0 = \begin{bmatrix} z_1 & z_2 & z_2 & z_1 \\ -z_2^* & z_1^* & z_1^* & -z_2^* \end{bmatrix}, \tag{2}$$

which is a QOD of rate 1 that transmits 2 complex symbols in 2 time-slots by two dual-polarized transmit antennas and has been listed in [4]. Moreover, it is easy to verify that it satisfies basic constraint needed for decoupled decoding given in [9], therefore, decoupled decoding solution can be easily obtained for it.

1b. Non-Permuted Designs: In this construction, the matrix $\mathbf{B}$ does not come from the permutation of the matrix $\mathbf{A}$, however, this category of symmetric-paired designs may or may not satisfy $\mathbf{C}^H\mathbf{C} = \lambda\mathbf{E}$. For that reason, it fails to provide decoupled decoding for some of QODs [8]. However, with our new proposed channel realization, it is shown that the solution is fully decoupled at the receiver. Our first example of this sub-category is given as

$$\mathbf{Q}_1 = \begin{bmatrix} z_1 + z_3 j & z_2 \\ -z_2^* & z_1^* + z_3 j \\ -z_3^* + z_1^* j & -z_2 j \\ z_2^* j & -z_3^* + z_1 j \end{bmatrix}. \tag{3}$$

This QOD is of rate $3/4$ as it transmits 3 complex symbols in 4 time-slots by two dual-polarized transmit antennas. Moreover, it is easy to verify that it satisfies basic complex orthogonality constraint $\mathbf{C}^H\mathbf{C} = \lambda\mathbf{I}$ and therefore, provide decoupled decoding. However, there can be other designs of this sub-category which may not provide decoupled decoding. To illustrate this point, we provide three QODs based on this approach. A QOD in which two independent CODs are employed to effectively transmit 4 complex symbols $\{z_1, z_2, z_3, z_4\}$ in 4 time-slots, i.e.,

$$\mathbf{Q}_2 = \mathbf{A} + \mathbf{B}j = \begin{bmatrix} z_1 + z_3 j & z_2 + z_4 j \\ z_2^* + z_4^* j & -z_1^* - z_3^* j \\ z_3 + z_1 j & z_4 + z_2 j \\ z_4^* + z_2^* j & -z_3^* - z_1^* j \end{bmatrix}, \tag{4}$$

which has rate 1. Interestingly, a non-trivial extension of standard Alamouti scheme in the quaternion domain can be found using real variables. Indeed such a design was formed in [8] with the combination of five reals $\{x_0, x_1, x_2, x_3, x_4\}$, providing us five complex numbers $z_1 = x_3 + x_4 i, z_2 = x_0 + x_2 i, z_3 = x_1 + x_2 i, z_4 = -x_1 + x_2 i, z_5 = x_0 - x_2 i$, encoded in a QOD of order 2

$$\mathbf{Q}_3 = \begin{bmatrix} z_1 + z_2 j & z_1 + z_3 j \\ z_1 + z_4 j & -z_1 + z_5 j \end{bmatrix}, \tag{5}$$

which has rate 5/4. Note that $z_4 = -z_3^*$ and $z_5 = z_2^*$, which clearly limits its realization in encoding. In Section IV, we will show how the above quaternion designs also achieve decoupled decoding. Another example of this construction is given below that provides a code rate of 2,

$$\mathbf{Q}_4 = \begin{bmatrix} z_1 + z_3 j & z_2 + z_4 j \\ z_2^* - z_4^* j & -z_1^* + z_3^* j \end{bmatrix}. \quad (6)$$

2) Non-Symmetric-Paired Designs. Non-symmetric-paired designs include other construction techniques of QODs which employ existing CODs to construct QODs but do not rely solely on symmetry property. An example of this category is quaternion permutation matrices-based QOD construction technique elaborated extensively in [4].

B. Quaternion-Based QODs

Pure quaternion-based construction techniques have the potential to increase the code rates of the coded matrices as COD-based designs often limit the code rate because of their complex orthogonality constraints. In literature, this construction technique has not been explored in detail because it entails stronger conditions to realize orthogonality. However, we would include the following pure quaternion-based QOD example, given in [4], which is based on two quaternion elements a and b

$$\mathbf{Q}_5 = \begin{bmatrix} a & jb \\ ib & -ka \end{bmatrix}. \quad (7)$$

This QOD satisfies quaternion orthogonality given that $ab^Q = ba^Q$. To realize this constraint, we assume that $a = b = x_0 + x_1 i + x_2 j + x_3 k$, where x_0, x_1, x_2 and x_3 denote real numbers. However, this assumption limits the rate of this QOD to 1.

III. System Model

We consider N_t dual-polarized transmit antennas and one dual-polarized receive antenna. The transmitting antennas transmit the coded sequences, where the channel matrix is represented as $\mathbf{H}^{(m)} = \begin{bmatrix} h_{11}^{(m)} & h_{12}^{(m)} \\ h_{21}^{(m)} & h_{22}^{(m)} \end{bmatrix}$, where $m = \{1, 2, ..., N_t\}$. The complex channel gains, $h_{12}^{(m)}$ and $h_{21}^{(m)}$, represent the cross polar scattering and the channel is assumed to be Rayleigh fading, which implies that each element of channel gain matrix is a complex Gaussian random variable (RV) with zero mean and unit variance. For this $N_t \times 1$ antenna arrangement, the received signal $\mathbf{R}$ can be obtained by applying inverse operator $\mathcal{C}^{-1}$ on

$$\mathcal{C}(\mathbf{R}) = \mathcal{C}(\mathbf{Q}) \begin{bmatrix} \mathbf{H}^{(1)} \\ \vdots \\ \mathbf{H}^{(N_t)} \end{bmatrix} + \begin{bmatrix} n_{11} & n_{12} \\ \vdots & \vdots \\ n_{N_t 1} & n_{N_t 2} \end{bmatrix} = \mathcal{C}(\mathbf{Q})\mathbf{H} + \mathbf{N}, \quad (8)$$

where $n_{k_1 k_2}\ \forall\ k_1 = \{1, 2, ..., N_t\}, k_2 = \{1, 2\}$ represent the entries of white noise as two dimensional independent and identically distributed (i.i.d.) complex Gaussian RVs with zero mean and identical variance per dimension, and $\mathbf{R}$ denotes a matrix containing received quaternion numbers. The complex code matrix $\mathcal{C}(\mathbf{Q})$ contains the transmitted symbols.

A. Problem Statement

It is proposed in [4] that a decoupled decoding can be obtained for any QOD even if the channel gain matrix of a dual-polarized transmission system is not modeled by a single quaternion gain. Later on, they corrected their decoding rule and clarified that the decoupled decoding can be achieved for certain QODs only. However, the properties of those QODs are not elaborated in their work. We would like to highlight that while investigating the ML-decoding criterion we are confronted with three main terms

$$\|\mathbf{R} - \mathcal{C}^{-1}\{\mathbf{CH}\}\|^2 = tr\big(\mathbf{R}^Q\mathbf{R}\big) - 2\Re\Big(tr\big(\mathbf{R}^Q\mathcal{C}^{-1}\{\mathbf{CH}\}\big)\Big) + \\ + tr\Big(\big(\mathcal{C}^{-1}\{\mathbf{CH}\}\big)^Q\big(\mathcal{C}^{-1}\{\mathbf{CH}\}\big)\Big), \quad (9)$$

of which the first term is a constant while the second term is already decoupled. The last term, however, can be simplified to $tr\Big(\mathbf{H}^H\mathbf{C}^H\mathbf{C}\mathbf{H}\Big)$, which in the case of symmetric-paired quaternion designs gets fully decoupled due to the identity that $\mathbf{C}^H\mathbf{C} = \lambda\mathbf{E}$.

Remark 1. It is evident from the simplification of the last term that it is still dependent on the complex orthogonality constraint $\mathbf{C}^H\mathbf{C}$ and therefore, the property $\mathbf{Q}^Q\mathbf{Q} = \lambda\mathbf{I}$ of QODs is not directly applicable to the quaternion ML norm.

Remark 2. The existence of identity $\mathbf{C}^H\mathbf{C} = \lambda\mathbf{E}$, in the case of symmetric-paired designs indicates that QODs are also effective in case of single-polarized antenna arrangements, which is not a surprise. This observation can also be verified through the simulation results of [5]-[6], in which the diversity order of the proposed 2×2 QOD is exactly same as that of 2×4 CODs.

It is clear from the above discussion that the operation $\mathcal{C}$, which transforms a QOD into a complex matrix is indeed causing the main problem. Besides the channel matrix is also complex which is why it is not possible to fully decouple the main term in ML-norm. On the other hand, it is easy to realize that one possible remedy of the problem is to convert either one or both of them into quaternion domain. In particular, the term $(\mathbf{QH})^Q = \mathbf{H}^Q\mathbf{Q}^Q$, will precisely give us a decoupled solution, in sharp contrast to $\mathcal{C}(\mathbf{Q})$ which fails to yield it. Below, we propose one possible physical realization to achieve this phenomenon.

B. Proposed Solution

We observe that to employ $\mathbf{Q}^Q\mathbf{Q} = \lambda\mathbf{I}$ in ML decoding rule, we need a redefined channel matrix based on the dual-polarized transmission channel at the receiver end as illustrated in Fig. 1. To realize this change of channel matrix, we precode the QOD to modify the transmitted signal in such a way that the desired channel coefficients embedded signals are received.

We now illustrate the significance in modifying the channel coefficients to resolve the issue of coupled decoding in QODs. For a given QOD $\mathbf{Q}$ of order $T \times N_t$, the channel gains can be realized in terms of quaternion as follows

$$\tilde{\mathbf{H}} = [\tilde{\mathbf{H}}^{(1)} \quad \tilde{\mathbf{H}}^{(2)} \cdots \tilde{\mathbf{H}}^{(N_t)}]^T, \quad (10)$$

where $\tilde{\mathbf{H}}^{(m)} = \big[h_{11}^{(m)} + h_{12}^{(m)} j\big]$, therefore, order of $\tilde{\mathbf{H}}$ is $N_t \times 1$. Channel gains in terms of quaternions are realizable in the form of (10) given that in the original channel matrix $\mathbf{H}^{(m)}$, $h_{21}^{(m)} = -h_{12}^{*(m)}$ and $h_{22}^{(m)} = h_{11}^{*(m)}$. This provides the required decoupled solution as given in the following theorem. For clarity and without the loss in generality, we omit the superscript (m) from the channels.

Theorem 3.1 *For dual-polarized antenna configurations, the transmission channel can be realized in terms of quaternionic channel gains (10) which decouples the term* $tr\big(\mathcal{C}^{-1}(\mathbf{CH})^Q\mathcal{C}^{-1}(\mathbf{CH})\big)$*, in (9) for all the transmitted symbols.*

Proof. It is easy to see that $\mathcal{C}^{-1}(\mathbf{CH})$, is equal to the product $\mathbf{Q}\tilde{\mathbf{H}}$, which was not possible earlier and consequently we get

$$\begin{aligned}\mathcal{C}^{-1}(\mathbf{CH})^Q\mathcal{C}^{-1}(\mathbf{CH}) =&(\mathbf{Q}\tilde{\mathbf{H}})^Q(\mathbf{Q}\tilde{\mathbf{H}}) = \tilde{\mathbf{H}}^Q\mathbf{Q}^Q\mathbf{Q}\tilde{\mathbf{H}},\\ =&\lambda\tilde{\mathbf{H}}^Q\mathbf{I}\tilde{\mathbf{H}} = \lambda\tilde{\mathbf{H}}^Q\tilde{\mathbf{H}}, \end{aligned} \quad (11)$$

which completes the proof. □

Provided the significance of the transformation of the channel matrix required at the receiver, we vary the amplitude and the phase of each transmitted symbol depending on its position in the coding matrix. We assume that perfect channel state information (CSI) is available at both the transmitter and the receiver.

In order to find the weight vector of complex numbers encoded in a QOD, a simple mechanism can be developed based on the following result. We then briefly describe its implementation through a relatively simple example.

Lemma 3.2. *The weight vector* $[w_{z_1}\ w_{z_2}]^T$*, of two complex symbols* z_1 *and* z_2 *transmitted at some instant through a dual polarized antenna, is given by*

$$\begin{bmatrix} w_{z_1} \\ w_{z_2} \end{bmatrix} = \begin{bmatrix} z_1h_{11} & z_2h_{21} \\ z_1h_{12} & z_2h_{22} \end{bmatrix}^{-1} \begin{bmatrix} z_1h_{11} - z_2h_{12}^* \\ z_1h_{12} + z_2h_{11}^* \end{bmatrix}. \quad (12)$$

Proof. To find the weight vector for a quaternion symbol q_1 having $\mathcal{C}^{-1}(q_1) = [z_1 \quad z_2]$ to be transmitted through dual-polarized antenna 1, we equate the required $[z_1 \quad z_2]\begin{bmatrix} h_{11}^{(1)} & h_{12}^{(1)} \\ -h_{12}^{*(1)} & h_{11}^{*(1)} \end{bmatrix}$ to its actual (but containing weighted transmitted complex symbols) $\mathbf{CH}$, i.e., $[z_1w_{z_1} \quad z_2w_{z_2}]\mathbf{H}^{(1)}$, where w_{z_1} and w_{z_2} are the weight coefficients. Eventually, we obtain a system of two complex algebraic equations to be solved for two unknowns. As the determinant of coefficient matrix is non-zero, therefore, a unique solution (12) is attained. □

The Equation (12) can provide the following generalized weight coefficients w_{z_1} and w_{z_2} to any quaternion symbol $q = z_1 + z_2j$

$$w_{z_1} = 1 - \frac{(h_{21}^{(t)}h_{11}^{*(t)} + h_{22}^{(t)}h_{12}^{*(t)})z_2}{(h_{11}^{(t)}h_{22}^{(t)} - h_{12}^{(t)}h_{21}^{(t)})z_1}, \quad (13)$$

$$w_{z_2} = \frac{h_{11}^{(t)}h_{11}^{*(t)} + h_{12}^{(t)}h_{12}^{*(t)}}{h_{11}^{(t)}h_{22}^{(t)} - h_{12}^{(t)}h_{21}^{(t)}}, \quad (14)$$

where the superscript t of the channel coefficients represents the dual-polarized antenna number through which the corresponding quaternion symbol q will be transmitted. It is important to note that besides the channel coefficients, complex symbols z_1 and z_2 also contribute to the weight coefficients. Consequently, the conventional system model (15) gets changed to the following form,

$$\mathbf{R} = \tilde{\mathbf{C}}\mathbf{H} + \mathbf{N}, \quad (15)$$

where $\tilde{\mathcal{C}}$ is the weighted complex code matrix of the QOD $\mathbf{Q}$.

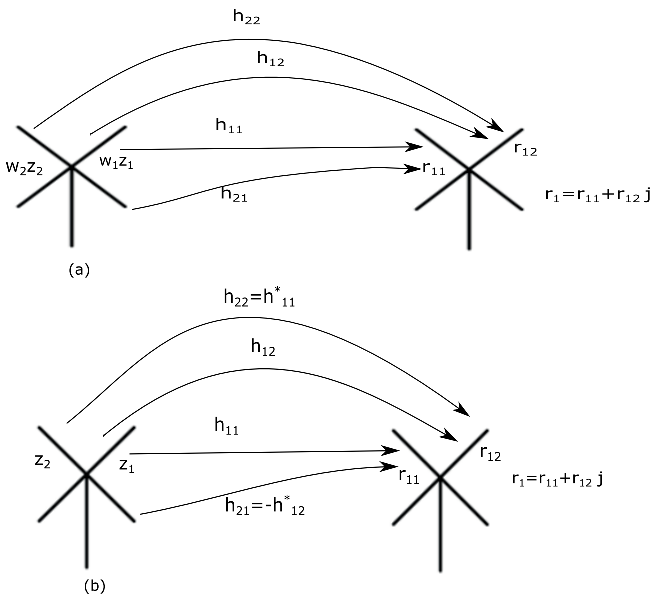

Figure 1: Two different realizations of the proposed system model, where (a) and (b) are equivalent to one another and provide the same received vector r_1.

IV. Proposed Low-Complexity Decoder

We employ ML decoder based on quaternion norm criterion, i.e., $\min_{z_u}\big(\|\mathbf{R} - \mathcal{C}^{-1}\{\mathbf{CH}\}\|^2\big)$, to decode the coded matrix $\mathbf{C}$ at the receiver end. However, we know that with the proposed system model, $\tilde{\mathbf{C}}\mathbf{H}$ gets changed to $\mathbf{Q}\tilde{\mathbf{H}}$ at the receiver side. The squared norm criterion (9), upon using the identity (11), finally assumes a decoupled form given by

$$\|\mathbf{R} - \mathcal{C}^{-1}\{\mathbf{CH}\}\|^2 = -2\Re\big(tr(\mathbf{R}^Q\mathbf{Q}\tilde{\mathbf{H}})\big) + \lambda tr\Big(\tilde{\mathbf{H}}^Q\tilde{\mathbf{H}}\Big). \quad (16)$$

It is important to note that the actual channel matrix is $\mathbf{H}$ but we employ $\tilde{\mathbf{H}}$ at the receiver. Moreover, it can be easily verified that for all the QODs presented in Section II, the conventional quaternion-norm-based ML norm criterion fails to provide decoupled decoding rule because $(\mathcal{C}(\mathbf{Q})^H\mathcal{C}(\mathbf{Q})) \neq \lambda\mathbf{I}$, for these QODs except $\mathbf{Q}_0$ and $\mathbf{Q}_1$. Now, we apply this proposed decoder on the examples presented in Section II to highlight that it works for all categories of the QODs. Here we use short hand notation for channel coefficients as $h^{(m)} = h_{11}^{(m)} + h_{12}^{(m)}j$ and $\gamma_1 = \sum_{k=1}^{2}\|h_{11}^{(k)}\|^2 + \|h_{12}^{(k)}\|^2$. As decoupled decoding solution of $\mathbf{Q}_0$ is already provided in [4] and [9], therefore, we begin with $\mathbf{Q}_1$.

Corollary 4.1. *The decoupled decoding statistics for each transmitted symbols* z_1, z_2 *and* z_3 *for QOD* $\mathbf{Q}_1$ *is*

$$\min_{z_1}\,(1+\gamma_1)|z_1|^2 - 2\Re\{r_1^Q z_1h^{(1)} + r_2^Q z_1^*h^{(2)} + r_3^Q z_1^*jh^{(1)} + r_4^Q z_1jh^{(2)}\},$$

$$\min_{z_2}\,(1+\gamma_1)|z_2|^2 - 2\Re\{r_1^Q z_2h^{(2)} - r_2^Q z_2^*h^{(1)} - r_3^Q z_2jh^{(2)} + r_4^Q z_2^*jh^{(1)}\},$$

$$\min_{z_3}\,(1+\gamma_1)|z_3|^2 - 2\Re\{r_1^Q z_3jh^{(1)} + r_2^Q z_3jh^{(1)} - r_3^Q z_3^*h^{(1)} - r_4^Q z_3^*h^{(2)}\}.$$

ISBN 978-3-8007-4560-9

Corollary 4.2. *The transmitted symbols z_1, z_2, z_3 and z_4 in QOD $\mathbf{Q}_2$ can be decoded using*

$$\min_{z_1} (1+\gamma_1)|z_1|^2 - 2\Re\{r_1^Q z_1 h^{(1)} - r_2^Q z_1^* h^{(2)} + r_3^Q z_1 j h^{(1)} - r_4^Q z_1^* j h^{(2)}\},$$
$$\min_{z_2} (1+\gamma_1)|z_2|^2 - 2\Re\{r_1^Q z_2 h^{(2)} - r_2^Q z_2^* h^{(1)} + r_3^Q z_2 j h^{(2)} + r_4^Q z_2^* j h^{(1)}\},$$
$$\min_{z_3} (1+\gamma_1)|z_3|^2 - 2\Re\{r_1^Q z_3 j h^{(1)} - r_2^Q z_3^* j h^{(2)} + r_3^Q z_3 h^{(1)} - r_4^Q z_3^* h^{(2)}\},$$
$$\min_{z_4} (1+\gamma_1)|z_4|^2 - 2\Re\{r_1^Q z_4 j h^{(2)} + r_2^Q z_4^* j h^{(1)} + r_3^Q z_4 h^{(2)} + r_4^Q z_4^* h^{(1)}\}.$$

Unlike the decoding of QODs $\mathbf{Q}_1$ and $\mathbf{Q}_2$, in which we decode complex numbers, the same can not be done for QOD $\mathbf{Q}_3$. It is not a surprise that QOD $\mathbf{Q}_3$, contains complex numbers of which two are conjugates which make it difficult to be extracted individually. However, the decoupled decoding of the same QOD is obtainable for the reals on which those complex symbols were constructed.

Corollary 4.3. *For QOD $\mathbf{Q}_3$, the simplified ML norm-based decoder for each real transmitted symbols x_0, x_1, x_2, x_3 and x_4 is*

$$\min_{x_0} (1+\gamma_1)|x_0|^2 - 2\Re\{r_1^Q x_0 j h^{(1)} + r_2^Q x_0 j h^{(2)}\},$$
$$\min_{x_1} (1+\gamma_1)|x_1|^2 - 2\Re\{r_1^Q (x_1 j h^{(2)}) + r_2^Q (-x_1 j h^{(1)})\},$$
$$\min_{x_2} (1+\gamma_1)|x_2|^2 - 2\Re\{r_1^Q x_2 k (h^{(1)} + h^{(2)}) + r_2^Q x_2 k (h^{(1)} - h^{(2)})\},$$
$$\min_{x_3} (1+\gamma_1)|x_3|^2 - 2\Re\{r_1^Q x_3 (h^{(1)} + h^{(2)}) + r_2^Q x_3 (h^{(1)} - h^{(2)}\},$$
$$\min_{x_4} (1+\gamma_1)|x_4|^2 - 2\Re\{r_1^Q x_4 i (h^{(1)} + h^{(2)}) + r_2^Q x_4 i (h^{(1)} - h^{(2)}\}.$$

Corollary 4.4. *The decoupled ML norm for each transmitted symbols z_1, z_2, z_3 and z_4 for QOD $\mathbf{Q}_4$ is*

$$\min_{z_1} (1+\gamma_1)|z_1|^2 - 2\Re\{r_1^Q z_1 h^{(1)} - r_2^Q z_1^* h^{(2)}\},$$
$$\min_{z_2} (1+\gamma_1)|z_2|^2 - 2\Re\{r_1^Q z_2 h^{(2)} + r_2^Q z_2^* h^{(1)}\},$$
$$\min_{z_3} (1+\gamma_1)|z_3|^2 - 2\Re\{r_1^Q z_3 j h^{(1)} + r_2^Q z_3^* j h^{(2)}\},$$
$$\min_{z_4} (1+\gamma_1)|z_4|^2 - 2\Re\{r_1^Q z_4 j h^{(2)} - r_2^Q z_4^* j h^{(1)}\}.$$

As in the case of QOD $\mathbf{Q}_3$, the decoupled decoding for QOD $\mathbf{Q}_5$ is only obtainable for real numbers only.

Corollary 4.5. *The decoupled ML norm for each real transmitted symbols x_0, x_1, x_2 and x_3 of a quaternion symbols $a = x_0 + x_1 i + x_2 j + x_3 k$ for QOD $\mathbf{Q}_5$ is*

$$\min_{x_0} (1+\gamma_1)|x_0|^2 - 2\Re\{r_1^Q (x_0 h^{(1)} + x_0 j h^{(2)}) + r_2^Q (x_0 i h^{(1)} - x_0 k h^{(2)})\},$$
$$\min_{x_1} (1+\gamma_1)|x_1|^2 - 2\Re\{r_1^Q (x_1 i h^{(1)} - x_1 k h^{(2)}) + r_2^Q (-x_1 h^{(1)} - x_1 j h^{(2)})\},$$
$$\min_{x_2} (1+\gamma_1)|x_2|^2 - 2\Re\{r_1^Q (x_2 j h^{(1)} - x_2 h^{(2)}) + r_2^Q (x_2 k h^{(1)} + x_2 i h^{(2)})\},$$
$$\min_{x_3} (1+\gamma_1)|x_3|^2 - 2\Re\{r_1^Q (x_3 k h^{(1)} + x_3 i h^{(2)}) + r_2^Q (-x_3 j h^{(1)} + x_3 h^{(2)}\}.$$

V. Simulation Results

For simulations, we used quadrature phase shift keying (QPSK) constellation for the transmission of data. Each transmitted complex coded matrix $\mathbf{C}$ is normalized obeying the constraint $E\|\mathbf{C}\|_F^2 = (2N_t)T$, where E represents the expectation, T denotes the number of timeslots and $\|\mathbf{C}\|_F^2$ is the squared Frobenious norm of the complex code matrix $\mathbf{C}$. Furthermore, white noise is added in each polarization independently. We have applied coupled ML quaternion norm and decoupled decoding rule on all the QODs presented in Section II to obtain the bit error rate (BER) curves. The constructed QODs $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3, \mathbf{Q}_4$ and $\mathbf{Q}_5$ provide $3/4, 1, 5/4, 2$ and 1 code rates for two dual-polarized transmit antenna arrangements, respectively. This implies that a code rate of 1 or more can be obtained with the help of quaternion orthogonality without compromise on system complexity for 4 single-polarized or 2 dual-polarized antenna arrangements.

Fig. 1 demonstrates the optimal performance of the proposed generalized low-complexity ML decoder for different categories of QODs. It shows the coupled and decoupled decoding rule match for different QODs, i.e., $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3, \mathbf{Q}_4$ and $\mathbf{Q}_5$, presented in Section II. It is important to note that only $\mathcal{C}(\mathbf{Q}_1)$ is a valid COD and $\mathcal{C}(\mathbf{Q}_2), \mathcal{C}(\mathbf{Q}_3), \mathcal{C}(\mathbf{Q}_4)$ and $\mathcal{C}(\mathbf{Q}_5)$ are quasi-orthogonal STBCs. Despite this comprise on complex orthogonality, we have obtained decoupled linear equation-based decoding solutions for all these QODs. Furthermore, $\mathbf{Q}_1$, i.e., a pure COD-based QOD, outperforms all other QODs for 2 transmit and 1 receive dual-polarized antenna-based arrangements from diversity perspective. However, as the code rate and number of transmitted symbols of these QODs are different, therefore, it would not be a fair comparison in terms of diversity. Overall, these curves demonstrate the phenomenon of diversity-multiplexing tradeoff, i.e., diversity of coded matrices decreases with the increase in the code rate. Fig. 1 also illustrates the efficiency of pure QOD designs as pure QOD-based design $\mathbf{Q}_5$ of code rate 1 provide the same diversity in two timeslots as compare to COD-based non-permuted design $\mathbf{Q}_2$ of the same code rate that uses 4 time slots. This implies that $\mathbf{Q}_5$ has half the decoding delay, i.e., the number of timeslots receiver has to wait to receive a complete coded matrix, as compare to $\mathbf{Q}_2$.

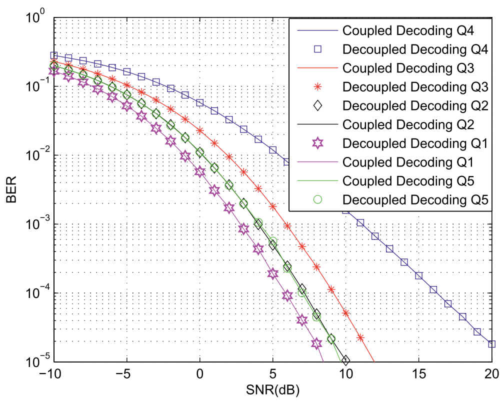

Figure 2: BER vs. SNR comparison of the ML coupled and decoupled decoding for different QODs for 2 transmit and 1 received dual-polarized antenna arrangement.

VI. Conclusion

We provided a solution to the issue of decoupled decoding by providing a generalized low-complexity decoder that can work for any QOD. This implies that a decoupled decoding solution can be obtained even for quasi-orthogonal or non-orthogonal coding matrices provided that they satisfy only the basic orthogonality constraint of QOD. We presented different QODs examples and applied the proposed generalized low-complexity decoder on them to validate its optimality.

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ISBN 978-3-8007-4560-9