This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the IEEE Globecom 2010 proceedings.

Modeling of a Cooperative One-Dimensional Multi-Hop Network using Quasi-Stationary Markov Chains

Syed Ali Hassan and Mary Ann Ingram
School of Electrical and Computer Engineering
Georgia Institute of Technology, Atlanta, Georgia 30332–0250
Email: alihassan@gatech.edu; mai@ece.gatech.edu

***Abstract*—We consider the irreducible discrete time Markov chain, with one absorbing state, as a potential candidate to model a wireless multi-hop transmission system that does cooperative transmission at every hop. This paper describes the modeling for a special geometry where the nodes are aligned on a one-dimensional horizontal grid with equal spacing and such that the cooperating clusters are adjacent. This model can be considered a precursor to a model for an Opportunistic Large Array broadcast for the finite density case. Assuming all the nodes have equal transmit power, the successive transmissions can be modeled as a Markov chain in discrete time. We derive the transition probability matrix of the Markov chain based on the hypoexponential distribution of the received power at a given time instant. The Perron-Frobenius eigenvalue of that sub-stochastic matrix is used in formulating a bound on how far transmissions can reach with a particular relay transmit power.**

I. Introduction

Wireless multi-hop communications, where radios forward the packets of other radios, has a wide variety of applications, not only in the cellular and sensor networking regimes, but in technologies like wireless computer networking and mobile computing. One promising, very fast, and energy efficient wireless transmission technique is the Opportunistic Large Array (OLA) [1], in which all radios that decode a message relay the message immediately after reception, without co-ordination with other relays. Because inter-node coordination is not needed, OLAs are particularly well suited for mobile networks, such as large groups of people with smart phones. For example, an OLA broadcast may complement or supplant Base Station or Access Point transmissions, harnessing the other radios in a network to increase the reliability and speed of a broadcast. OLA broadcasting is also a likely candidate for wireless sensor networks, a key technology in pervasive computing. There are many uncertainties that influence exactly which radios participate in an OLA. Currently, there is no way to model general OLA transmissions short of brute force Monte Carlo simulation, and this has been a barrier to fundamental analysis of this transmission technique, when the density of the nodes is finite.

In this paper, we model a variant of a *decode and forward* OLA network, where the nodes are uniformly spaced along a line. Furthermore, we constrain the OLAs to be contained in a pre-specified non-overlapping sets of nodes. Each OLA is still opportunistic in the sense that only the nodes in the set that can decode will be part of the OLA. Also, we assume that the distance between the source and the destination is long enough that the transmission reaches a kind of steady state. Specifically, we assume that the conditional probability that the kth node in cluster decodes, given that the previous cluster had at least one node transmitting, is the same for each cluster. This allows us to apply the well-established theory of quasi-stationary discrete time Markov chains with one absorbing state [5]. The absorbing state is defined to be when all the nodes in one hop cannot decode the message, and the transmissions stop propagating.

Most of the previous works in cooperative transmission deal with a single [2] or dual relay system [3]. The authors in [4] studied large dense networks, using the *continuum* assumption. Under this assumption, the density of nodes goes to infinity while the total relay power is kept fixed. This assumption is not appropriate for low-density networks. The continuum model was also used in [1], where the authors studied broad-casting and uni-casting protocols with the pathloss as the only channel impairment. Thus in this paper, we remove the continuum assumption by deploying a simple one-dimensional network where the nodes are uniformly spaced on a grid. We also assume Rayleigh flat fading channels in addition to the pathloss as the channel characteristic and derive the transition probability matrix of the Markov chain.

The rest of the paper is organized as follows. In the next section, we describe the network layout along with its parameters. Section III deals with the proposed modeling of the network via discrete time Markov chains (DTMC) and obtaining a quasi-stationary distribution of this chain. In section IV, we analytically derive the transition probability matrix for the proposed model and then we will validate the analytical results with those of numerical simulations in Section V. The paper then concludes with certain recommendations of the future work in Section VI.

II. System Description

Consider an infinite line of nodes where adjacent nodes are a distance d apart from one another as shown in Figure 1. Each hop or decoding level is characterized by a fixed number, M, of nodes that are eligible to cooperate to send the

This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the IEEE Globecom 2010 proceedings.

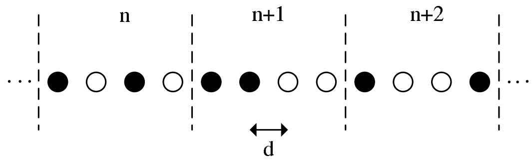

Fig. 1. Arrangement of nodes on a grid, $M = 4$

same message signal to the next level. A node, at any certain level, can decode and forward (DF) the message without error, when its received signal-to-noise ratio (SNR), from previous level only, is greater than or equal to a modulation dependent threshold, τ. Example DF nodes are shown as filled black circles in Figure 1. We assume that all the nodes transmit with the same transmit power P_t. A node, while receiving, receives copies of the message signal from the nodes that decoded the message correctly in the previous level, over orthogonal fading channels using equal gain combining (EGC). Let us define $\mathbb{N}_n = \{m_1(n), m_2(n), ..., m_{k_n}(n)\}$, where k_n is the cardinality of the set $\mathbb{N}_n$, to be the set of indices of those nodes that decoded the signal perfectly at the time instant (or hop) n. For example, from Figure 1, $\mathbb{N}_n = \{1, 3\}$ and $\mathbb{N}_{n+1} = \{1, 2\}$. The received power at the jth node at the next time instant $n + 1$ is given by

$$Pr_j(n+1) = \frac{P_t}{d^\beta} \sum_{m \in \mathbb{N}_n} \frac{\mu_{mj}}{(M - m + j)^\beta}, \quad (1)$$

where the summation is over the DF nodes in the previous level. μ_{mj} is the flat fading Rayleigh channel gain from node m in the previous level to node j in the current level. The elements of μ are independently and identically distributed (i.i.d) and are drawn from an exponential distribution with the parameter σ_μ^2=1, and β is the pathloss exponent with a usual range of 2-4. Consequently, the received SNR at the jth node is given as $\gamma_j = Pr_j/\sigma_j^2$, where σ_j^2 is the variance of noise at the jth receiver. Throughout the paper, we will use the notation $Pr_j(n)$ as the power received at the jth node at the nth time instant. We assume perfect timing and frequency recovery at each receiver, and we also assume that there is sufficient transmit synchronization between the nodes of a level, such that all the nodes in a level transmit to the next level at the same time [7]. In other words, the transmissions only occur at discrete instants of time $n, n+1, ...$ such that the hop number and the time instants can be defined by just one index n.

III. Modeling by Markov Chain

At a certain time n, each node from the nth level will take part in the next transmission, if it has decoded the data perfectly, or it will not take part, if it did not decode correctly. The decisions of all the nodes in a level can be represented as $X(n) = [\mathbb{I}_1(n), \mathbb{I}_2(n), ..., \mathbb{I}_M(n)]$, where $\mathbb{I}_j(n)$ is the indicator random variable for the jth node at the nth time instant given as

$$\mathbb{I}_j(n) = \left\{ \begin{array}{ll} 1 & \text{node } j \text{ decodes} \\ 0 & \text{node } j \text{ does not decode} \end{array} \right\} \quad (2)$$

Thus each node is represented by either 0 or 1 depending upon the successful decoding of the received data. For example, from Figure 1, we have $\mathbb{I}_1(n) = \mathbb{I}_3(n) = 1$ and $\mathbb{I}_2(n) = \mathbb{I}_4(n) = 0$. We observe that the outcomes of $X(n)$ are M-bit binary words, each outcome constituting a state, and there are 2^M number of states, which are enumerated in decimal form $\{0, 1, ..., 2^M - 1\}$. Let i_n be the outcome at time n. For example, $i_n = \{1010\}$ in binary, and $i_n = 10$ in decimal in Figure 1. Then we may write

$$\mathbb{P}\{X(n) = i_n | X(n-1) = i_{n-1}, ..., X(1) = i_1\} = \mathbb{P}\{X(n) = i_n | X(n-1) = i_{n-1}\} \quad (3)$$

where $\mathbb{P}$ indicates the probability measure. Equation (3) implies that $X(n)$ is a discrete time finite state Markov Process. Assuming the statistics of the channel are same for all the hops in the network, the Markov chain can be regarded as a homogeneous one.

It can be further noticed that at any point in time, there is a probability that the Markov chain can go into an absorbing state, thus terminating the transmission. That will be a state when all the nodes at a particular hop cannot decode the message perfectly and thus Markov chain will be in the 0 state (decimal). Thus we consider the Markov chain, X, on a state space $\{0\} \cup S$, where S is a finite transient irreducible state space, $S = \{1, 2, ..., 2^M - 1\}$, and 0 being the absorbing state where

$$\lim_{n \to \infty} \mathbb{P}\{X(n) = 0\} \nearrow 1 \quad \text{a.s.} \quad (4)$$

On the other hand, the rest of the $2^M - 1$ states make an irreducible state space, i.e. there is always a probability to go from any transient state to another transient state. We will define two matrices to describe the Markov Chain. The first, $\tilde{\mathbf{P}}$, is the full, $2^M \times 2^M$ transition probability matrix for the states in the set $\{0\} \cup S$. Each row in $\tilde{\mathbf{P}}$ sums to one. The second matrix, $\mathbf{P}$, in our description, is the submatrix of $\tilde{\mathbf{P}}$ that is formed by striking the column and row that involve transitions to and from the 0 state. Therefore, $\mathbf{P}$ is a $(2^M - 1) \times (2^M - 1)$ matrix corresponding to the states in S. It can be noticed that the transition probability matrix $\mathbf{P}$ on the state space S is not right stochastic, i.e. the row entries of $\mathbf{P}$ do not sum to 1 because of the killing probabilities given as

$$\kappa_i = 1 - \sum_{j \in S} \mathbf{P}_{ij}, \qquad i \in S. \quad (5)$$

Since $\mathbf{P}$ is a square irreducible nonnegative matrix, then by the Perron-Frobenius theorem [9] there exists an eigenvalue, ρ, such that following properties are true:

- $\rho \geq |r|$ for any eigenvalue $r \neq \rho$.
- The eigenvector associated with ρ is unique and has strictly positive entries.

For proof, please refer to [9] and [11]. Since $\mathbf{P}$ is not right stochastic, thus $\rho < 1$. Also since all states in S are transient and not strictly self-communicating, thus $\rho > 0$ [6]. Overall our assumptions imply that

$$0 < \rho < 1. \quad (6)$$

From the theory of Markov chains [11], we know that a distribution $\mathbf{u} = (u_i, i \in S)$ is called ρ-invariant distribution if $\mathbf{u}$ is the left eigenvector of the transition matrix $\mathbf{P}$ corresponding to the eigenvalue ρ, i.e.

$$\mathbf{uP} = \rho\mathbf{u}. \tag{7}$$

We are now interested in the limiting behavior of this Markov chain as time proceeds. Since $\forall n$, $\mathbb{P}\{X(n) = 0\} > 0$, thus eventual killing is certain. But we are interested in finding the distribution of the transient states, before the killing occurs. The so-called limiting distribution is called the quasi-stationary distribution of the Markov chain, which is independent of the initial conditions of the process. From [5] and [6], this unique distribution is given by the ρ-invariant distribution for one step transition probability matrix of the Markov chain on S. We can find the quasi-stationary distribution by getting the *maximum* eigenvector, $\hat{\mathbf{u}}$ of $\mathbf{P}$, then defining $\mathbf{u} = \hat{\mathbf{u}}/\sum_{i=1}^{2^M-1} \hat{\mathbf{u}}_i$ as a normalized version of $\hat{\mathbf{u}}$ that sums to one.

Thus we can define the unconditional probability of being in state j as

$$\mathbb{P}\{X(n) = j\} = \rho^n u_j, \quad j \in S, \quad n \geq 0. \tag{8}$$

We also let $T = \inf\{n \geq 0 : X(n) = 0\}$ denote the end of survival time, i.e. the time at which killing occurs. It follows then,

$$\mathbb{P}\{T > n + m | T > n\} = \rho^m, \tag{9}$$

while the quasi-stationary distribution of the Markov chain is given as

$$\lim_{n\to\infty} \mathbb{P}\{X(n) = j | T > n\} = u_j, \quad j \in S. \tag{10}$$

IV. Formulation of the Transition Probability Matrix

In this section, we will find the state transition matrix $\mathbf{P}$ for our model, the eigenvector of which will give us the quasi-stationary distribution. Let i and j denote a pair of states of the system such that $i, j \in \{1, 2, ..., 2^M - 1\}$, where each i and j are the decimal equivalent of the binary word formed by the set of indicator random variables. Now for each node m, the probability of being able to decode at time n is given as

$$\begin{aligned}\mathbb{P}\{\text{node } m \text{ of level } n \text{ will decode}\} =& \mathbb{P}\{\mathbb{I}_m(n) = 1\} \\ =& \mathbb{P}\{\gamma_m(n) > \tau\}.\end{aligned} \tag{11}$$

Similarly, the probability of outage or the probability of $\mathbb{I}_m(n) = 0$ is given as $1 - \mathbb{P}\{\gamma_m(n) > \tau\}$ where

$$\mathbb{P}\{\gamma_m(n) > \tau\} = \int_\tau^\infty p_{\gamma_m}(y)dy. \tag{12}$$

$p_{\gamma_m}(y)$ is the PDF of the received SNR at the mth node. It can be seen that the power at a certain node is the sum of the finite powers from the previous level nodes, each of which is exponentially distributed. Thus for K independently distributed exponential random variables with respective parameters λ_k, where $k = 1, 2, ..., K$, the resulting distribution of the sum of these random variables is known as Hypoexponential distribution [8] which is given as

$$p_Y(y) = \sum_{k=1}^K C_k \lambda_k \exp(-\lambda_k y), \tag{13}$$

where

$$C_k = \prod_{\zeta \neq k} \frac{\lambda_\zeta}{\lambda_\zeta - \lambda_k}. \tag{14}$$

Although $\int_0^\infty p_Y(y)dy = 1$, it should not be thought that C_k are probabilities, because some of them will be negative. Thus from (12), the probability of success of the mth node is given as

$$\begin{aligned}\mathbb{P}\{\gamma_m(n) > \tau | X(n-1) \in S\} =& \sum_{k=1}^M C_k \exp\left(-\lambda_k^{(m)}\tau\right) \\ & \times \mathbb{I}_k(n-1),\end{aligned} \tag{15}$$

where C_k is as given in (14) and $\lambda_k^{(m)}$ is given as

$$\lambda_k^{(m)} = \frac{d^\beta (M - k + m)^\beta \sigma_m^2}{P_t}. \tag{16}$$

Equation (15) describes the probability of success of one node. For M nodes in a level, let us define the index sets corresponding to the ith state as

$$\mathbb{N}_n^{(i)} = \left\{m_1^{(i)}(n), m_2^{(i)}(n), ..., m_{k_n}^{(i)}(n)\right\}$$

and

$$\bar{\mathbb{N}}_n^{(i)} = \{1, 2, ..., M\} \backslash \mathbb{N}_n^{(i)}.$$

So the one step transition probably for going from state i to j is given as

$$\begin{aligned}P_{ij} =& \prod_{k\in\mathbb{N}_{n+1}^{(j)}} \left\{\sum_{m\in\mathbb{N}_n^{(i)}} C_m \exp\left(-\lambda_m^{(k)}\tau\right)\right\} \\ & \prod_{k\in\bar{\mathbb{N}}_{n+1}^{(j)}} \left\{1 - \sum_{m\in\mathbb{N}_n^{(i)}} C_m \exp\left(-\lambda_m^{(k)}\tau\right)\right\}\end{aligned} \tag{17}$$

V. Numerical Results

In this section, we compare the analytical results with those of numerical simulations for different sets of parameters. For the purpose of simulations, we calculate the received power at each node based on the previous state (assuming an initial distribution of nodes at the first hop), which is used to set the indicator functions as either 1 or 0 depending upon the threshold criterion. These indicator functions will form the current state and the process continues. We finally obtain the distribution of the chain by simulating over 20,000 trials. The Perron-Frobenius eigenvalue of $\mathbf{P}$ has been found using [10]. In all of the following results, we set $P_t = 1$, $\sigma^2 = 1$, $\sigma_\mu^2 = 1$, $d = 0.5$ and $\beta = 2$.

This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the IEEE Globecom 2010 proceedings.

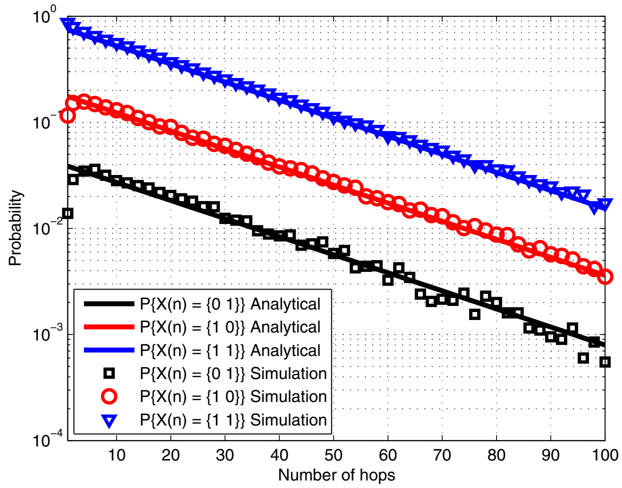

Fig. 2. Distribution of the states for M=2

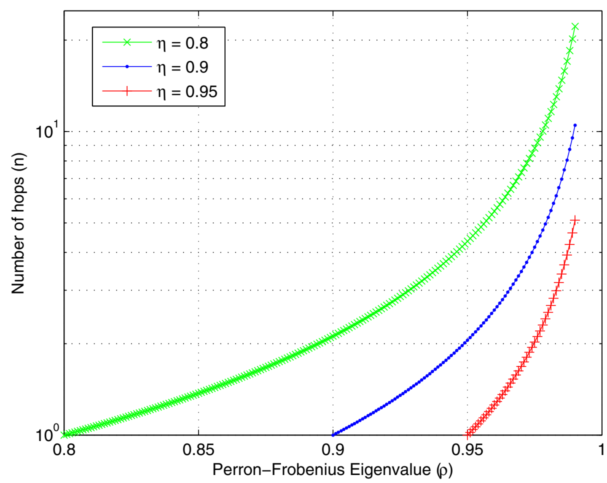

Fig. 4. Relationship between hops and ρ

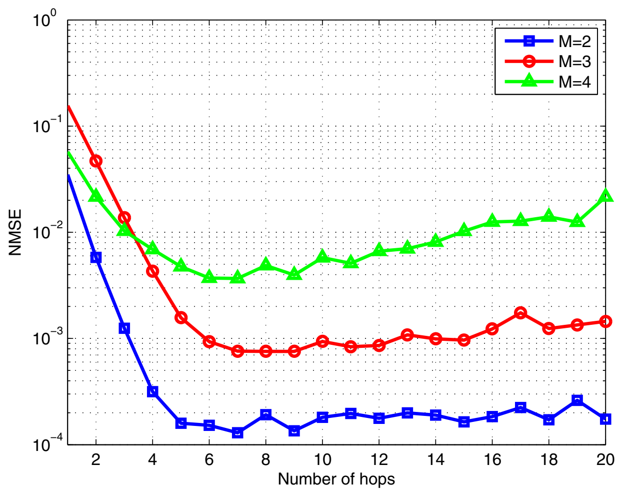

Fig. 3. NMSE between the quasi-stationary distributions from analysis and simulations

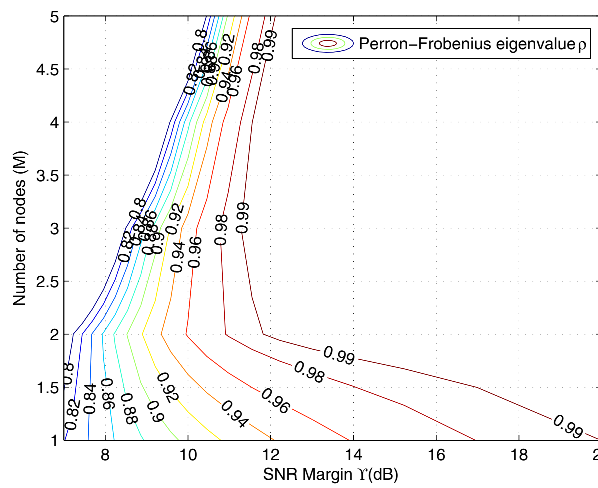

Fig. 5. Contours of ρ as a function of nodes, M, and Υ

Figure 2 shows the distribution of the Markov chain, at different hops, when there are $M = 2$ nodes in each hop. Thus it can be seen that the analytical results are quite close to that of simulations. It can be further noticed that as we increase the hop number, the probability of being in a transient state decreases which asserts the relationship as described in (4). Figure 3 shows the normalized mean squared error (NMSE) between the quasi-stationary distribution for different values of M, where the NMSE is defined as

$$NMSE = \frac{1}{2^M - 1} \frac{\|\mathbf{u} - \hat{\mathbf{u}}\|_2^2}{< \mathbf{u} >^2} \quad (18)$$

where $\hat{\mathbf{u}}$ is the quasi-stationary distribution obtained from simulations, $||.||_2^2$ is the squared Euclidean norm and $< . >$ is the mean value of the vector. The figure shows that as we increase the hop number, we approach the quasi-stationary distribution quite fast. As we increase M, the NMSE starts to increase and these deviations in the numerical and analytical results can be attributed as the precision errors while calculating the eigenvalues of larger matrices.

From the deployment perspective of the network, it is sometimes desirable to optimize the values of certain parameters like transmit power of relays or distance between them. This optimization is done to obtain a certain quality of service η. In other words, we are interested in finding the probability of delivering the message at a certain distance without having entered the 0 state, and we desire this probability to be at least η where $\eta \sim 1$ ideally. Thus (9) gives us a nice upper bound on the value of m (the number of hops) one can go with a given η, i.e. $\rho^m \geq \eta$, which gives

$$m \leq \frac{\ln \eta}{\ln \rho}. \quad (19)$$

978-1-4244-5638-3/10/$26.00 ©2010 IEEE

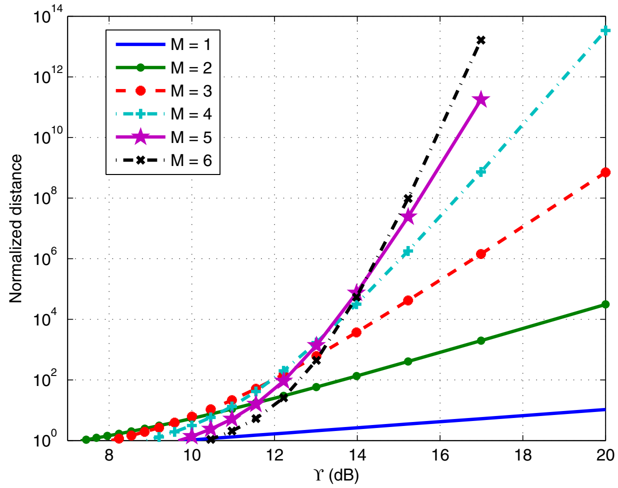

Fig. 6. SNR margin required to reach a particular distance for various values of M with $\eta = 0.9$

This bound has been plotted in Figure 4 for various values of η. Thus if the destination is far off, we require more hops, m, which will require a larger value of ρ. Now ρ is a nonlinear function of M, P_t, σ^2, d, and β. An exact analytical expression of ρ in terms of theses parameters is difficult to obtain but from (15) and (16), we define

$$\Upsilon = \frac{\gamma_0}{\tau} = \frac{P_t}{d^\beta \sigma^2}\frac{1}{\tau}, \tag{20}$$

as the SNR margin from a single transmitter d distance away. A large SNR margin corresponds to a large node degree, whereas an SNR margin of 1 implies a node degree of exactly two in this line-network. We can find the values of ρ for different Υ and M. The contours of ρ for various values of M and Υ are plotted in Figure 5. Thus for a fixed η, we can find the ρ from (19) or Figure 4, as $\rho \geq \exp \frac{\ln \eta}{m}$, and from Figure 5, we can look at what values of M and Υ correspond to this Perron eigenvalue. It can be noticed that if we do not have cooperation, i.e. single node in each hop, M=1, then additional 2-8dB of SNR margin is required to deliver the same message to a destination, than with cooperation. However, Figure 5 does not address fixed distance and it is hard to determine the diversity gain, since the distance is increasing as M increases.

Figure 6 shows the relationship between required SNR margin to reach the destination node at a particular normalized distance for different values of nodes in each hop. The normalized distance, which is the true distance divided by d, is defined as the product of the number of hops, (made to reach the destination) and the number of nodes, M, in each hop. We observe that, for SNR margin (Υ) greater than about 14dB, as we increase M, we can go farther for the same SNR margin. It can be further noticed that the curves intersect each other at some points, which shows that the transmissions can reach a particular point in two ways, i.e. keeping M small and increasing the number of hops or increasing M and having a smaller number of hops. In the context of a 2-D network, the former enables more simultaneous routes in the network while the later improves the latency of a given route.

VI. Conclusions and Recommendations

In this paper, we have shown that a one-dimensional multi-hop network that does opportunistic large array transmission can be modeled as a Markov chain in discrete time and we derived the sub-stochastic matrix of this chain. The Perron-Frobenius eigenvalue of this matrix helps in determining different parameters for achieving better performance in delivering the message to a destination. As an extension to this work, it is recommended to obtain a framework where the number of nodes, M, can be opportunistic, i.e. the OLA boundaries should not be fixed, and they should dynamically grow or shrink depending upon the random nature of data received at different locations.

References

[1] L. Thanayankizil, A. Kailas, and M.A. Ingram, "Routing for Wireless Sensor Networks with an Opportunistic Large Array (OLA) Physical Layer," *Ad Hoc and Sensor Wireless Networks, Special Issue on the 1st International Conference on Sensor Technologies and Applications*, Vol. 8, No. 1-2, pp. 79-117, 2009.

[2] J.N. Laneman, D.N.C. Tse, and G.W. Wornell, "Cooperative diversity in wireless networks: Efficient protocols and outage behavior," *IEEE Trans. Inf. Theory*, vol. 50, pp. 3062-3080, Dec. 2004.

[3] S.A. Hassan, Y.G. Li, P.S.S. Wang, and M. Green, "A full rate dual relay cooperative approach for wireless systems, " *KICS/IEEE Journal of Commun. and Networks*, to appear.

[4] B.S. Mergen, A. Scaglione, and G. Mergen, "Asymptotic analysis of multi-stage cooperitve broadcast in wireless networks," *IEEE Trans. Inf. Theory*, vol. 52, No. 6, pp. 2531-2550, June 2006.

[5] J.N. Darroch and E.Seneta, "On quasi-stationary distributions in absorbing discrete-time finite Markov chains," *Journal of Applied Probability*, vol. 2, No. 1, pp. 88-100, June 1965.

[6] E.A. van Doorn and P.K. Pollett, "Quasi-stationary distributions for reducible absorbing Markov chains in discrete time," *Journal of Markov Processes and Related Fields*, vol. 15, No. 2, pp. 191-204, 2009.

[7] Y.J. Chang and M.A. Ingram, "Cluster transmission time synchronization for Cooperative transmission using software defined radio," in *IEEE (ICC) Workshop on Cooperative and Cognitive Mobile Networks (CoCoNet3)*, May 2010.

[8] S.M. Ross, *Introduction to Probability Models*, 9th Edition, Academic Press, 2007.

[9] C.D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM publishers, 2001.

[10] P. Chanchana, "An Algorithm for computing the Perron root of a nonnegative irreducible matrix," Ph.D. Dissertation, North Carolina State University, Raleigh, 2007

[11] E. Senta, *Non-Negative Matrices and Markov Chains*, 2nd edition, Springer New York, 2006.